**80-110 Nature of Mathematical Reasoning**

Spring 2002, *Dirk Schlimm*

HANDOUT #9

Wednesday, February 20, 2002

## 1. DAVID HILBERT ON THE SOLVABILITY IN PRINCIPLE OF EVERY MATHEMATICAL QUESTION

David Hilbert (1862–1943) was one of the most prominent mathematicians of the late 19th and early 20th century.

In 1900 Hilbert proposed his famous 23 problems in Paris, where he avowed to the conviction "that every definite mathematical problem must necessarily be susceptible of an exact settlement." This is a conviction, he said, "which every mathematician shares, although it has not yet been supported by proof." And, more definite: "in mathematics there is no *ignorabimus*."

However, 1918 he declared the solvability in principle of every mathematical question as a "difficult epistemological question" which is related to the question of the consistency of the integers and of sets, and which should be carefully investigated.

In Hilbert's biography we read: "In addition, two other motives were in opposition to each other—both strong tendencies in Hilbert's way of thinking. On one side, he was convinced of the soundness of existing mathematics; on the other side, he had—philosophically—a strong scepticism."

"The problem of Hilbert," Bernays [Hilbert's assistant and collaborator] explains, "was to bring together these opposing tendencies, and he thought that he could do this through the method of formalizing mathematics." ([Reid 1970], p. 174.)

Later, in 1930, Hilbert emphatically claimed:

> "there are absolutely no unsolvable problems. [ . . . ]
> We must know, we shall know."

## 2. KURT GÖDEL (1906–1978)

The following is a direct quotation from the beginning of Gödel's famous essay.

“ON FORMALLY UNDECIDABLE PROPOSITIONS OF *PRINCIPIA MATHEMATICA* AND RELATED SYSTEMS I
(1931)
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The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia Mathematica (PM)* on the one hand and the Zermelo-Fraenkel axiom system of set theory (further developed by J. von Neumann) on the other. These two systems are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide *any* mathematical question that can at all be formally expressed in these systems. It will be shown below that this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integers that cannot be decided on the basis of the axioms. This situation is not in any way due to the special nature of the systems that have been set up but holds for a wide class of formal systems; among these, in particular, are all systems that result from the two just mentioned through the addition of a finite number of axioms, provided no false propositions of the kind specified in footnote 4 become provable owing to the added axioms.”

## 3. Gödel's incompleteness theorems

‘PA’ stands for ‘Peano arithmetic’, i.e., the theory of the natural numbers based on the axioms of Dedekind (1888) and Peano (1889).

- **Gödel's first incompleteness theorem:** If PA is consistent, then there are arithmetical statements $\gamma$ such that neither $\gamma$ nor $\neg\gamma$ is provable in PA; i.e. PA is incomplete.

- **Gödel's second incompleteness theorem:** If PA is consistent, then it cannot prove its own consistency, i.e., $PA \nvdash Cons_{PA}$.