80-110 Nature of Mathematical Reasoning
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— **Handout #15 (4/17/02)** —

Important definitions: Sets & Functions

Set: A *set* A is a collection of objects, called *elements* of A. To say that a is an element of A we write, $a \in A$. To define a set S of all those things x that satisfy a certain property P, we write: $S = \{x|P(x)\}$ (see also FOL, p. 210).

Function:

- **Algorithmic.** A *function* is an algorithm or recipe according to which to every element of the set A exactly one value from the set B is assigned.

- **Set theoretical.** A function $F : A \rightarrow B$ from set A (the *domain*) to set B (the *range*) is
  1. a set of *ordered pairs* ($< a, b >$) of which the first element is in A and the second in B, ($F = \{< a_1, b_1 >, < a_2, b_2 >, \dots\}$).
  2. where all elements of A occur as the first element of one such ordered pair,
  3. and to every element of A there is exactly one element of B.

  Formally,
  $$F = \{< a, b > \mid (a \in A \;\&\; b \in B) \;\& \\ (\forall a \in A \; \exists b \in B \quad < a, b > \in F) \;\& \\ (\forall < a_1, b_1 >, < a_2, b_2 > \in F \quad [a_1 = a_2 \rightarrow b_1 = b_2]).$$
  Note that in this definition we rely only on the language of first-order logic and the predicates $\in$ and $=$.

1-1 function: A function $F_{1-1} : A \rightarrow B$ is *1-1, (one-to-one, injective)*, if and only if the function does not map two different elements in the domain to the same element in the range: $\forall < a_1, b_1 >, < a_2, b_2 > \in F_{1-1} \quad [a_1 = a_2 \leftrightarrow b_1 = b_2]$.

Cardinality: The *cardinality* of a set A, written $|A|$, is the number of elements of A.

$|A| \leq |B|$: If A and B are sets, then $|A| \leq |B|$ if and only if there is a 1-1 function from A to B ("no double-counting").

$|A| = |B|$: If A and B are sets, then $|A| = |B|$ if and only if $|A| \leq |B|$ and $|B| \leq |A|$.

$\aleph_0$, denumerable, countable: The cardinality of the natural numbers $\mathbb{N}$ is $\aleph_0$ (aleph nought, aleph is the first letter in the Hebrew alphabet), i.e., $|\mathbb{N}| = \aleph_0$. Any set with cardinality $\aleph_0$ is said to be *denumerable* or *countable*.