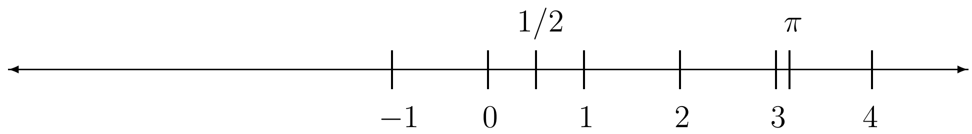
Classes of Real Numbers

All real numbers can be represented by a line:



The Real Line

$$\text{real numbers}\begin{cases}\text{rational numbers}\begin{cases}\text{integers}\\ \text{non-integral fractions}\end{cases}\\ \text{irrational numbers}\end{cases}$$

Rational numbers
All of the real numbers which consist of a ratio of two integers.
Irrational numbers
Most real numbers are **not** rational, i.e. there is no way of writing them as the ratio of two integers. These numbers are called **irrational**.
Familiar examples of irrational numbers are: $\sqrt{2}$, π and e.

How to represent numbers?

- The **decimal**, or **base 10**, system requires 10 symbols, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.
- The **binary**, or **base 2**, system is convenient for electronic computers: here, every number is represented as a string of **0**'s and **1**'s.

Decimal and binary representation of **integers** is simple, requiring an expansion in nonnegative powers of the base; e.g.

$$(71)_{10} = 7 \times 10 + 1$$

and its **binary equivalent:** $(1000111)_2 =$

$$1 \times 64 + 0 \times 32 + 0 \times 16 + 0 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1.$$

Non-integral **fractions** have entries to the **right** of the point. e.g. **finite** representations

$$\frac{11}{2} = (5.5)_{10} = 5 \times 1 + 5 \times \frac{1}{10},$$

$$\frac{11}{2} = (101.1)_2 = 1 \times 4 + 0 \times 2 + 1 \times 1 + 1 \times \frac{1}{2}$$

Infinitely Long Representations

But 1/10, with finite **decimal** expansion $(0.1)_{10}$, has the **binary** representation

$$\begin{aligned}\frac{1}{10} &= (0.0001100110011\ldots)_2 \\ &= \frac{1}{16}+\frac{1}{32}+\frac{0}{64}+\frac{0}{128}+\frac{1}{256}+\frac{1}{512}+\cdots.\end{aligned}$$

This, while **infinite**, is **repeating**.
1/3 has **both** representations infinite *and* repeating:

$$1/3 = (0.333\ldots)_{10} = (0.010101\ldots)_2.$$

If the representation of a **rational** number is *infinite*, it **must** be *repeating*. e.g.

$$1/7 = (0.142857142857\ldots)_{10}.$$

Irrational numbers always have **infinite, non-repeating** expansions. e.g.

$$\begin{aligned}\sqrt{2} &= (1.414213...)_{10}, \\ \pi &= (3.141592\ldots)_{10}, \\ e &= (2.71828182845\ldots)_{10}.\end{aligned}$$

Converting between binary & decimal.

- **Binary $\longrightarrow$ decimal**:
  Easy. e.g. $(1001.11)_2$ is the decimal number

  $$1\times 2^3+0\times 2^2+0\times 2^1+1\times 2^0+1\times 2^{-1}+1\times 2^{-2}=9.75$$

- **Decimal $\longrightarrow$ binary**:
  Convert the integer and fractional parts separately.
  e.g. if x is a **decimal integer**, we want coefficients $a_0, a_1, \ldots, a_n$, all 0 or 1, so that

  $$a_n\times 2^n + a_{n-1}\times 2^{n-1}+\cdots+a_0\times 2^0 = x,$$

  has representations $(a_n a_{n-1}\cdots a_0)_2 = (x)_{10}$.

  Clearly dividing x by 2 gives **remainder** a_0, leaving as **quotient**

  $$a_n\times 2^{n-1}+a_{n-1}\times 2^{n-2}+\cdots+a_1\times 2^0,$$

  and so we can continue to find a_1 then a_2 etc.

  Q: What is a similar approach for decimal fractions?

Computer Representation of Numbers

- **Integers** — three ways:

  1. **sign-and-modulus** — a simple approach.

     Use 1 bit to represent the **sign**, and store the **binary** representation of the magnitude of the integer. e.g. decimal 71 is stored as the bitstring

0	00...01000111

     If the computer **word size** is 32 bits, $2^{31}-1$ is the **largest** magnitude which will fit.

  2. **2's complement representation (CR)**
     more convenient, & used by most machines.

     (i) The **nonnegative** integers 0 to $2^{31}-1$ are stored as before, e.g., 71 is stored as the bitstring

     $$\boxed{000\ldots01000111}$$

     (ii) A **negative integer** $-x$, where $1 \le x \le 2^{31}$, is stored as the **positive integer** $2^{32}-x$.

     e.g. -71 is stored as the bitstring

     $$\boxed{111\ldots10111001}.$$

     Converting x to its 2's CR $2^{32}-x$ of $-x$:

     $$2^{32}-x = (2^{32}-1-x)+1,$$

     $$2^{32}-1 = (111\ldots111)_2.$$

     Chang all zero bits of x to ones, one bits to zero and adding one.

     Q: *What is the quickest way of deciding if a number is negative or nonnegative using 2's CR* **??**

     An advantage of 2's CR:
     Form $y+(-x)$, where $0 \le x, y \le 2^{31}-1$.

     $$\text{2's CR of } y \text{ is } y\,; \quad \text{2's CR of } -x \text{ is } 2^{32}-x$$

     Adding these two **representations** gives

     $$y+(2^{32}-x) = 2^{32}+y-x = 2^{32}-(x-y).$$

– If $y \geq x$, the LHS will not fit in a 32-bit word, and the **leading bit** can be dropped, giving the **correct result**, $y - x$.

– If $y < x$, the RHS is **already correct**, since it *represents* $-(x - y)$.

Thus, **no special hardware is needed for integer subtraction**. The addition hardware can be used, once $-x$ has been represented using 2's complement.

3. **1's complement representation:**
a negative integer $-x$ is stored as $2^{32} - x - 1$.
This system *was* used, but no longer.

- **Non-integral real numbers.**

  Real numbers are approximately stored using the **binary** representation of the number.

  Two possible methods:
  . **fixed point** and **floating point**.

  Fixed point: the computer word is divided into **three fields**, one for each of:
  • the **sign** of the number
  • the number **before** the point
  • the number **after** the point.

  In a **32-bit word** with field widths of **1,15** and **16**, the number 11/2 would be stored as:

0	000000000000101	1000000000000000

  The fixed point system has a severe limitation on the **size** of the numbers to be stored. e.g.

  Q: smallest to **largest** magnitudes above?

  Inadequate for most scientific computing.

Normalized Exponential Notation

In (normalized) **exponential notation**,
a nonzero real number is written as

$$\pm m \times 10^E, \qquad 1 \leq m < 10,$$

- m is called the **significand** or mantissa,
- E is an **integer**, called the **exponent**.

For the **computer** we need **binary**, write $x \neq 0$ as

$$x = \pm m \times 2^E, \quad \text{where } 1 \leq m < 2.$$

The binary expansion for m is

$$m = (b_0.b_1b_2b_3\ldots)_2, \quad \text{with } b_0 = 1.$$

IEEE Floating Point Representation

Through the efforts of **W. Kahan** & others, a **binary** floating point standard was developed: IEEE 754-1985. It has now been adopted by almost all computer manufacturers. Another standard, IEEE 854-1987 for radix independent floating point arithmetic, is devoted to both binary (radix-2) and decimal (radix- 10) arithmetic. The current version is IEEE 754-2008, including nearly all of the original IEEE 754-1985 and IEEE 854-1987

We write **IEEE FPS** for the binary standard.

Three important requirements:

- **consistent** representation of floating point numbers across machines
- **correctly rounded** arithmetic
- **consistent** and **sensible** treatment of exceptional situations (e.g. division by 0).

IEEE Single format

There are **3 standard types** in IEEE FPS: **single**, **double**, and **extended** format.

Single format numbers use **32-bit words**.

A 32-bit **word** is divided into **3 fields**:

- **sign field**: **1** bit (**0** for positive, **1** for negative).
- **exponent field**: **8** bits for E.
- **significand field**: **23** bits for m.

In the **IEEE single format system**,
the **23** significand bits are used to store $b_1b_2\ldots b_{23}$.

Do not store b_0, since we know $b_0 = 1$. This idea is called **hidden bit normalization**.

The **stored bitstring** $b_1b_2\ldots b_{23}$ is now the **fractional part** of the significand, the significand field is also referred to as the **fraction field**.

It may not be possible to store x with such a scheme, because

- either E is outside the permissible range (see later).
- or $b_{24}, b_{25}, \ldots$ are **not all zero**.

Def. A number is called a **(computer) floating point number** if it can be stored **exactly** this way. e.g

$$71 = (1.000111)_2 \times 2^6$$

can be represented by

0	$E = 6$	00011100000000000000000

.

If x is not a floating point number, it must be **rounded** before it can be stored on the computer.

Special Numbers

- 0. Zero cannot be **normalized**.

  A pattern of **all 0s** in the fraction field of a normalized number represents the significand **1.0**, not **0.0**.
- -0. -0 and 0 are **two different representations for the same value**
- ∞. This allows e.g. $1.0/0.0 \to \infty$, instead of terminating with an **overflow** message.
- $-\infty$. $-\infty$ and ∞ represent **two very different numbers**.
- NaN, or **"Not a Number"**, and is an **error pattern**.
- Subnormal numbers (see later)

All **special numbers** are represented by **a special bit pattern** in the **exponent field**.

Precision, Machine Epsilon

Def. Precision :
The number of bits in the significand (including the hidden bit) is called the **precision** of the floating point system, denoted by p.
In the **single format** system, $p = 24$.
Def. Machine Epsilon:
The gap between the number **1** and the **next larger** floating point number is called the **machine epsilon** of the floating point system, denoted by ϵ.
In the **single format** system, the number after 1 is

$$b_0.b_1 \ldots b_{23} = 1.00000000000000000000001,$$

so $\epsilon = 2^{-23}$.

IEEE Single format

$\pm$	$a_1a_2a_3\ldots a_8$	$b_1b_2b_3\ldots b_{23}$

If exponent $a_1\ldots a_8$ is	Then value is
$(00000000)_2 = (0)_{10}$	$\pm(0.b_1..b_{23})_2 \times 2^{-126}$
$(00000001)_2 = (1)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{-126}$
$(00000010)_2 = (2)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{-125}$
$(00000011)_2 = (3)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{-124}$
$\downarrow$	$\downarrow$
$(01111111)_2 = (127)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{0}$
$(10000000)_2 = (128)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{1}$
$\downarrow$	$\downarrow$
$(11111100)_2 = (252)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{125}$
$(11111101)_2 = (253)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{126}$
$(11111110)_2 = (254)_{10}$	$\pm(1.b_1..b_{23})_2 \times 2^{127}$
$(11111111)_2 = (255)_{10}$	$\pm\infty$ if $b_1, , b_{23} = 0$; NaN otherwise.

The $\pm$ refers to the sign, **0** for **positive**, **1** for **negative**.

- All lines *except the first and the last* refer to the **normalized** numbers, i.e. **not special**.

  The exponent representation $a_1a_2\ldots a_8$ uses **biased representation**: this bitstring is the binary representation of $E + 127$. 127 is the **exponent bias**.
  e.g. $1 = (1.000\ldots0)_2 \times 2^0$ is stored as

0	01111111	00000000000000000000000

  Exponent range for normalized numbers is
  00000001 to 11111110 (1 to 254),
  representing **actual exponents**

  $$\boxed{E_{min} = -126 \text{ to } E_{max} = 127}$$

  The **smallest normalized positive number** is

  $$(1.000\ldots0)_2 \times 2^{-126} :$$

0	00000001	00000000000000000000000

  approximately 1.2×10^{-38}.

  The **largest normalized positive number** is

  $$(1.111\ldots1)_2 \times 2^{127} :$$

0	11111110	11111111111111111111111

  approximately 3.4×10^{38}.

- *Last line*:

If exponent $a_1 \ldots a_8$ is	Then value is
$(11111111)_2 = (255)_{10}$	$\pm\infty$ if $b_1, \ldots, b_{23} = 0$; NaN otherwise

This shows an **exponent bitstring of all ones** is a special pattern for $\pm\infty$ or NaN, depending on the value of the fraction.

- *First line*

$(00..00)_2 = (0)_{10}$	$\pm(0.b_1..b_{23})_2 \times 2^{-126}$

shows **zero** requires a zero bitstring for the *exponent* field **as well as** for the *fraction*:

0	00000000	00000000000000000000000

Initial unstored bit is 0, not 1, in line 1.

If **exponent** is **zero**, but **fraction** is **nonzero**, the number represented is **subnormal**.

Although 2^{-126} is the smallest normalized positive number, we can represent smaller numbers called **subnormal** numbers.
e.g. $2^{-127} = (0.1)_2 \times 2^{-126}$:

0	00000000	10000000000000000000000

and $2^{-149} = (0.0000\ldots01)_2 \times 2^{-126}$:

0	00000000	00000000000000000000001

This is the smallest positive number we can store.

Subnormal numbers cannot be normalized, as this gives exponents which do not fit.

Subnormal numbers are **less accurate**, (less room for nonzero bits in the fraction). e.g. $(1/10) \times 2^{-133} = (0.11001100\ldots)_2 \times 2^{-136}$ is

0	00000000	00000000011001100110011001

Q: IEEE single format:
(i) How is 2. represented **??**

0	10000000	00000000000000000000000

(ii) What is the next smallest IEEE single format number larger than 2 **??**

0	10000000	00000000000000000000001

(iii) What is the gap between 2 and the first IEEE single format number larger than 2?

$$2^{-23} \times 2 = 2^{-22}.$$

General Result:

Let $x = m \times 2^E$ be a normalized single format number, with $1 \le m < 2$. The **gap** between x and the next smallest single format number larger than x is

$$\epsilon \times 2^E.$$

IEEE Double format

Each **double format** floating point number is stored in a **64-bit double word**.
Ideas the same; field widths (1, 11 & 52) and exponent bias (1023) different. $b_1, .., b_{52}$ can be stored instead of $b_1, .., b_{23}$.

$\pm$	$a_1a_2a_3 \ldots a_{11}$	$b_1b_2b_3 \ldots b_{52}$

If exponent is $a_1..a_{11}$	Then value is
$(000..0000)_2 = (0)_{10}$	$\pm(0.b_1..b_{52})_2 \times 2^{-1022}$
$(000..0001)_2 = (1)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{-1022}$
$(000..0010)_2 = (2)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{-1021}$
$(000..0011)_2 = (3)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{-1020}$
$\downarrow$	$\downarrow$
$(01..111)_2 = (1023)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{0}$
$(10..000)_2 = (1024)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{1}$
$\downarrow$	$\downarrow$
$(11..100)_2 = (2044)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{1021}$
$(11..101)_2 = (2045)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{1022}$
$(11..110)_2 = (2046)_{10}$	$\pm(1.b_1..b_{52})_2 \times 2^{1023}$
$(11..111)_2 = (2047)_{10}$	$\pm\infty$ if $b_1,, b_{52} = 0$; NaN otherwise

Single versus Double

- Single format is required.
- Double format is optional, but is provided by almost all machines implementing the standard.
- Support for the requirements may be provided by **hardware** or **software**.
- But almost all machines have hardware support for both single and double format.

Extended format

IEEE FPS recommends support for extended format:

- **exponent**: at least 15 bits;
- **fractional part of the significand**: at least 63 bits.

Intel microprocessors:
80-bit word, with **1 bit sign**, **15 bit exponent** and **64 bit significand**, with leading bit of a normalized number **not** hidden.
The arithmetic is implemented in hardware.

Machine Epsilon of the 3 Formats

Single	$\epsilon = 2^{-23} \approx 1.2 \times 10^{-7}$
Double	$\epsilon = 2^{-52} \approx 2.2 \times 10^{-16}$
Extended (Intel)	$\epsilon = 2^{-63} \approx 1.1 \times 10^{-19}$

Significant Digits

- In single format, the significand of a normalized has 24 **bits of accuracy**. This corresponds to **approximately** 7 significant decimal digits, since

$$2^{-24} \approx 10^{-7}.$$

- In double format, the significand has 53 **bits of accuracy**. This corresponds to **approximately** 16 **significant decimal digits**, since

$$2^{-53} \approx 10^{-16}.$$

Rounding

We use **Floating Point Numbers (FPN)** to include

$$\pm 0, \text{ subnormal \& normalized FPNs, \& } \pm\infty$$

in a given format, e.g. single. These form a **finite** set. Notice **NaN** is not included.
N_{min}: the minimum positive **normalized** FPN;
N_{max}: the maximum positive **normalized** FPN;

A real number x is in the **"normalized range"** if

$$N_{min} \le |x| \le N_{max}.$$

Q: Let x be a real number and $|x| \le N_{\max}$. If x is <u>not</u> a floating point number, what are two obvious choices for the floating point **approximation** to x **??**

x_- the closest FPN **less** than x; x_+ the closest FPN **greater** than x.

Using **IEEE single** format, if x is positive with

$$x = (b_0.b_1b_2\ldots b_{23}b_{24}b_{25}\ldots)_2 \times 2^E,$$
$$b_0 = 1 \text{ (normalized)}, \text{ or } b_0 = 0,\ E = -126 \text{ (subnormal)}$$

then **discarding** b_{24}, b_{25}, ... gives.

$$x_- = (b_0.b_1b_2\ldots b_{23})_2 \times 2^E,$$

An algorithm for x_+ is more complicated since it may involve some bit "carries".

$$x_+ = [(b_0.b_1b_2\ldots b_{23})_2 + (0.00\ldots 1)_2] \times 2^E.$$

If x is **negative**, the situation is reversed: x_+ is obtained by dropping bits b_{24}, b_{25}, etc.

Correctly Rounded Arithmetic

The IEEE standard defines the **correctly rounded value of** x, $\boxed{\text{round}(x)}$.

If x **is** a floating point number, $\text{round}(x) = x$. Otherwise $\text{round}(x)$ depends on the **rounding mode** in effect:

- **Round down:** $\text{round}(x) = x_-$.
- **Round up:** $\text{round}(x) = x_+$.
- **Round towards zero:**
  $\text{round}(x)$ is either x_- or x_+,
  whichever is between zero and x.

- **Round to nearest:** round(x) is either x_- or x_+, whichever is nearer to x. In the case of a tie, the one with its **least significant bit equal to zero** is chosen.

  This rounding mode is almost always used.

If x is **positive**, then x_- is between zero and x, so **round down** and **round towards zero** have the same effect. **Round towards zero** simply requires **truncating** the binary expansion, i.e. discarding bits.
"Round" with no qualification usually means **"round to nearest"**.

Absolute Rounding Error

Def. The **absolute rounding error** associated with x:

$$|\text{round}(x) - x|.$$

Its value depends on mode.
For all modes $|\text{round}(x) - x| < |x_+ - x_-|$.
Suppose $N_{\min} \le x \le N_{\max}$,

$$x = (b_0.b_1b_2 \ldots b_{23}b_{24}b_{25}\ldots)_2 \times 2^E, \quad b_0 = 1.$$

$$\text{IEEE } \textbf{single} \quad x_- = (b_0.b_1b_2\ldots b_{23})_2 \times 2^E.$$

$$\text{IEEE } \textbf{single} \quad x_+ = x_- + (0.00\ldots 01)_2 \times 2^E.$$

So for **any** mode

$$|\text{round}(x) - x| < 2^{-23} \times 2^E.$$

In general for **any rounding mode**:

$$|\text{round}(x) - x| < \epsilon \times 2^E. \quad (*)$$

Q: (i) For **round towards zero**, could the absolute rounding error equal $\epsilon \times 2^E$ **??**
(ii) Does $(*)$ hold if $0 < x < N_{\min}$, i.e. $E = -126$ and $b_0 = 0$ **??**

Relative Rounding Error, $x \neq 0$

The **relative rounding error** is defined by $|\delta|$,

$$\delta \equiv \frac{\text{round}(x)}{x} - 1 = \frac{\text{round}(x) - x}{x}.$$

Assuming x is in the normalized range,

$$x = \pm m \times 2^E, \quad \text{where } m \geq 1,$$

so $|x| \geq 2^E$. Since $|\text{round}(x) - x| < \epsilon \times 2^E$, we have, for **all** rounding modes,

$$|\delta| < \frac{\epsilon \times 2^E}{2^E} = \epsilon. \quad (*)$$

Q: Does $(*)$ necessarily hold if $0 < |x| < N_{\min}$, i.e. $E = -126$ and $b_0 = 0$ **??** Why **??**
Note for any real x in the **normalized range**,

$$\boxed{\text{round}(x) = x(1 + \delta), \quad |\delta| < \epsilon.}$$

An Important Idea

From the definition of δ we see

$$\text{round}(x) = x(1 + \delta),$$

so the **rounded value** of an arbitrary number x in the **normalized range** is **equal to** $x(1+\delta)$, where, regardless of the rounding mode,

$$|\delta| < \epsilon.$$

This is very important, because you can think of the stored value of x as **not exact**, but as **exact within a factor** of $1 + \epsilon$.
IEEE **single format numbers** are good to a factor of about $1 + 10^{-7}$, which means that they **have about 7 accurate decimal digits**.

Special Case of Round to Nearest

For **round to nearest**, the **absolute** rounding error can be **no more than <u>half</u> the gap between x_- and x_+**. This means in IEEE single, for all $|x| \le N_{\max}$:

$$|\text{round}(x) - x| \le 2^{-24} \times 2^E,$$

and in general

$$|\text{round}(x) - x| \le \frac{1}{2}\epsilon \times 2^E.$$

The previous analysis for **round to nearest** then gives for x in the **normalized range**:

$$\text{round}(x) = x(1+\delta),$$

$$|\delta| \le \frac{\frac{1}{2}\epsilon \times 2^E}{2^E} = \frac{1}{2}\epsilon.$$

Operations on Floating Point Numbers

There are very few: standard **arithmetic** operations: $+$, $-$, $*$, $/$, plus **comparison**, **square root**, etc.

The **operands** must be available in the processor memory. But <u>combining</u> two floating point numbers may **not** give a floating point number, e.g. **multiplication** of two **24-bit** significands generally gives a **48-bit** significand.

When the result is **not** a floating point number, the <u>IEEE standard requires</u> that the computed result be the <u>**correctly** rounded value of the **exact** result</u>

Q: Using **round to nearest**, which of $1+10^{-5}$, $1+10^{-10}$, $1+10^{-15}$ round to 1 in IEEE single format and double format, respectively **??**

IEEE Rule for Rounding

Let x and y be floating point numbers, and let $\oplus$,$\ominus$,$\otimes$,$\oslash$ denote the **implementations** of $+$,$-$,$*$,$/$ on the computer. Thus $x \oplus y$ is the computer's **approximation** to $x+y$.

The **IEEE rule** is then precisely:

$$\begin{aligned} x \oplus y &= \text{round}(x+y), \\ x \ominus y &= \text{round}(x-y), \\ x \otimes y &= \text{round}(x \times y), \\ x \oslash y &= \text{round}(x/y). \end{aligned}$$

From our discussion of relative rounding errors, when $x+y$ is in the **normalized range**,

$$\boxed{x \oplus y = (x+y)(1+\delta), \quad |\delta| < \epsilon,}$$

for **all** rounding modes. Similarly for $\ominus$, $\otimes$ and $\oslash$.
Note that $|\delta| \le \epsilon/2$ for **round to nearest**.

Implementing Correctly Rounded FPS

Consider **adding** two IEEE single format FPNs:

$$x = m \times 2^E \quad \text{and} \quad y = p \times 2^F.$$

First (if necessary) **shift one significand**, so both numbers have the same **exponent** $G = \max\{E, F\}$. The significands are then **added**, and if necessary, the result **normalized** and **rounded**. e.g. adding $3 = (1.100)_2 \times 2^1$ to $3/4 = (1.100)_2 \times 2^{-1}$:

$$\begin{array}{rll} & (\ 1.10000000000000000000000 &)_2 \times 2^1 \\ + & (\ 0.01100000000000000000000 &)_2 \times 2^1 \\ = & (\ 1.11100000000000000000000 &)_2 \times 2^1. \end{array}$$

Further normalizing & rounding is not needed.
Now add 3 to 3×2^{-23}. We get

$$\begin{array}{rll} & (\ 1.10000000000000000000000 &)_2 \times 2^1 \\ + & (\ 0.00000000000000000000001|1 &)_2 \times 2^1 \\ = & (\ 1.10000000000000000000001|1 &)_2 \times 2^1 \end{array}$$

Result is **not** an IEEE single format FPN, and so must be **correctly rounded**.

Guard Bits

Correctly rounded floating point addition and subtraction is not trivial even the result is a FPN. e.g. $x - y$,

$$x = (1.0)_2 \times 2^0, \qquad y = (1.1111\ldots1)_2 \times 2^{-1}.$$

Aligning the significands:

$$\begin{array}{rll} & (\ 1.00000000000000000000000| &)_2 \times 2^0 \\ - & (\ 0.11111111111111111111111|1 &)_2 \times 2^0 \\ = & (\ 0.00000000000000000000000|1 &)_2 \times 2^0 \end{array}$$

an example of **cancellation** — most bits in the two numbers cancel each other.
The result is $(1.0)_2 \times 2^{-24}$, a floating point number, but to obtain this we must **carry out the subtraction using an extra bit**, called a **guard bit**. Without it, we would get a wrong answer.

More than one guard big may be needed. e.g., consider computing $x - y$ where $x = 1.0$ and $y = (1.000\ldots1)_2 \times 2^{-25}$. Using 25 guard bits:

$$
\begin{array}{rlr}
 & (\,1.00000000000000000000000| &)_2 \times 2^0 \\
- & (\,0.00000000000000000000000|0100000000000000000000001 &)_2 \times 2^0 \\
= & (\,0.11111111111111111111111|1011111111111111111111111 &)_2 \times 2^0 \\
\text{Normalize:} & (1.11111111111111111111111|0111111111111111111111110 &)_2 \times 2^{-1} \\
\text{Round to nearest:} & (1.11111111111111111111111 &)_2 \times 2^{-1}
\end{array}
$$

If we use only 2 guard bits, we will get a wrong result:

$$
\begin{array}{rlr}
(& \;\;1.00000000000000000000000| &)_2 \times 2^0 \\
-(& \;\;0.00000000000000000000000|01 &)_2 \times 2^0 \\
=(& \;\;0.11111111111111111111111|11 &)_2 \times 2^0 \\
\text{Normalize: }(& \;\;1.11111111111111111111111|1 &)_2 \times 2^{-1} \\
\text{Round to nearest: }(& 10.00000000000000000000000 &)_2 \times 2^{-1} \\
\text{Renormalize: }(& \;\;1.00000000000000000000000 &)_2 \times 2^0
\end{array}
$$

We would still get the same wrong result even if 3, 4, or as many as 24 guard bits.
Machines that implement correctly rounded arithmetic take such possibilities into account, and it turns out that **correctly rounded results can be achieved in all cases using only**

$$\text{two guard bits } + \text{ sticky bit,}$$

where the **sticky bit** is used to **flag** a rounding problem of this kind.

For the previous example, we use two guard bits and "turn on" a sticky bit to indicates that at least one nonzero extra bit was discarded when the bits of y were shifted to the right past the 2nd guard bit (the bit is called "sticky" because once it is turned on, it stays on, regardless of how many bits are discarded):

$$
\begin{array}{rlr}
 & (\,1.00000000000000000000000| &)_2 \times 2^0 \\
- & (\,0.00000000000000000000000|011 &)_2 \times 2^0 \\
= & (\,0.11111111111111111111111|101 &)_2 \times 2^0 \\
\text{Normalize:} & (\,1.11111111111111111111111|01 &)_2 \times 2^{-1} \\
\text{Round to nearest:} & (\,1.11111111111111111111111 &)_2 \times 2^{-1}
\end{array}
$$

Multiplication and Division

If $x = m \times 2^E$ and $y = p \times 2^F$, then

$$x \times y = (m \times p) \times 2^{E+F}$$

Three steps:

- **multiply** the significands,
- **add** the exponents,
- **normalize** and correctly **round** the result.

Division requires taking the quotient of the significands and the difference of the exponents.
Multiplication and **division** are substantially more complicated than **addition** and **subtraction**.
In principle it is possible, by using enough space on the chip, to implement the operations so that they are all **equally fast**.
In practice, chip designers build the hardware so that **multiplication** is approximately **as fast as addition**.
However, the **division** operation, the most complicated to implement, generally takes **significantly longer** to execute than **addition** or **multiplication**.

Exceptional Situations

When a reasonable response to exceptional data is possible, it should be used.
The simplest example is **division by zero**. Two **earlier** standard responses:

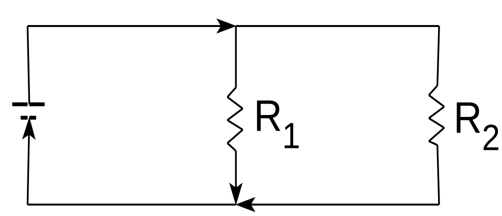
- generate the **largest FPN** as the result.

  Rationale: user would notice the large number in the output and conclude something had gone wrong.

  Disaster: e.g. $2/0 - 1/0$ would then have a result of 0, which is **completely meaningless**. In general the user might **not even notice** that any error had taken place.

- generate a **program interrupt**, e.g.
  "fatal error — division by zero".
  The burden was on the programmer to make sure that division by zero would **never** occur.

Example: Consider computing the **total resistance** in an electrical circuit with two resistors (R_1 and R_2 ohms) connected in parallel:



The formula for the **total resistance** is

$$T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

What if $R_1 = 0$? If one resistor offers no resistance, **all** the current will flow through that and avoid the other; therefore, the total resistance in the circuit is **zero**. The formula for T also makes perfect sense **mathematically**:

$$T = \frac{1}{\frac{1}{0} + \frac{1}{R_2}} = \frac{1}{\infty + \frac{1}{R_2}} = \frac{1}{\infty} = 0.$$

The IEEE FPS Solution

Why should a **programmer** have to worry about treating division by zero as an exceptional situation here?

In **IEEE floating point arithmetic**, if the initial floating point environment is set properly:

division by zero does not generate an interrupt

but **gives an infinite result**,

program execution continuing normally.

In the case of **the parallel resistance formula** this leads to a final **correct result** of $1/\infty = 0$, following the mathematical concepts **exactly**:

$$T = \frac{1}{\frac{1}{0} + \frac{1}{R_2}} = \frac{1}{\infty + \frac{1}{R_2}} = \frac{1}{\infty} = 0.$$

Other uses of ∞

We used some of the following:

$$\begin{aligned} a > 0 \ &: \ a/0 \to \infty \\ & \quad\ a * \infty \to \infty, \\ a \text{ finite} \ &: \ a + \infty \to \infty \\ & \quad\ a - \infty \to -\infty \\ & \quad\ a/\infty \to 0 \\ & \quad\ \infty + \infty \to \infty. \end{aligned}$$

But

$$\infty * 0, \quad 0/0, \quad \infty/\infty, \quad \infty - \infty$$

make no sense. Computing any of these is called an **invalid operation**, and the IEEE standard sets the result to NaN (**Not a Number**). **Any** arithmetic operation on a NaN **also** gives a NaN result.

Whenever a NaN is discovered in the output, the programmer **knows something has gone wrong**.

(An ∞ in the output may or may not indicate an error, depending on the context).

Overflow and Underflow

Overflow is said to occur when

$$N_{max} < |\text{ true result }| < \infty,$$

where N_{max} is the **largest** normalized FPN.

Two **pre-IEEE** standard treatments:

(i) Set the result to ($\pm$) N_{max}, or

(ii) Interrupt with an **error message**.

In IEEE arithmetic, the standard response depends on the **rounding mode**:

Suppose that the overflowed value is **positive**. Then

rounding model	result
round up	∞
round down	N_{max}
round towards zero	N_{max}
round to nearest	∞

Round to nearest is the **default** rounding mode and any other choice may lead to very misleading final computational results.

Underflow is said to occur when

$$0 < |\text{ true result }| < N_{min},$$

where N_{min} is the **smallest** normalized floating point number.

Historically the response was usually:

replace the result by zero.

In **IEEE arithmetic**, the result may be a **subnormal** number instead of zero. This allows results **much smaller** than N_{min}. But there may still be a significant loss of accuracy, since subnormal numbers have fewer bits of precision.

IEEE Standard Response to Exceptions

Invalid Opn.	Set result to NaN
Division by 0	Set result to $\pm\infty$
Overflow	Set result to $\pm\infty$ or $\pm N_{max}$
Underflow	Set result to ± 0, $\pm N_{\min}$ or subnormal
Inexact	Set result to correctly rounded value

- The IEEE FPS requires that an **exception** must be **signaled** by setting an associated **status flag**,
- The IEEE FPS highly recommends that the programmer should have the option of

  either **trapping the exception** — providing special code to be executed when the exception occurs,

  or **masking the exception** — the program continues with the response of the table.
- A high level language may not allow trapping.
- It is usually best to rely on these standard responses.

Floating Point in C

In C, the type **float** refers to a **single precision** floating point variable.
e.g. **read** in a floating point number, using the standard input routine `scanf`, and **print** it out again, using `printf`:

```
main ()     /* echo.c:  echo the input */
{
    float x;
    scanf("%f", &x);
    printf("x = %f", x);
}
```

The 2nd argument `&x` to `scanf` is the **address** of `x`. The routine `scanf` needs to know **where** to store the value read.
The 2nd argument `x` to `printf` is the **value** of `x`.
The 1st argument `"%f"` to both routines is a **control string**.
The two standard **formats** used for specifying floating point numbers in these control strings are:

- `%f` , for **fixed decimal** format
- `%e` , for **exponential decimal format**

`%f` and `%e` have **identical** effects in `scanf`, which can process input in a **fixed** decimal format (e.g. 0.666) or an **exponential** decimal format (e.g. `6.66e-1`, meaning 6.66×10^{-1}.
The following results were for a Sun 4.
With input 0.66666666666666666666 :

```
#cc -o echoinput echo.c
#echoinput
0.66666666666666666666      (typed in)
x= 0.666667                 (printed out)
```

Using Different Output Formats in `printf`

```
Output format      Output
    %f              0.666667
    %e              6.666667e-01
    %8.3f           0.667
    %8.3e           6.667e-01
    %20.15f         0.666666686534882
    %20.15e         6.666666865348816e-01
```

The input is correctly rounded to about 6 or 7 digits of precision, so `%f` and `%e` print, **by default**, 6 digits after the decimal point.

The next two lines print to **less** precision. The 8 refers to the **total** field width, the 3 to the number of digits **after** the point.

In the last two lines about half the digits have **no significance**.

Regardless of the **output format**, the floating point variables are **always** stored in the **IEEE formats**.

Double or **Long Float**

Double precision variables are declared in C using **double** or **long float**. But changing **float** to **double** above:

{`double x; scanf("%f",&x); printf("%e",x);`}

gives `-6.392091e-236`. **Q:** Why **??**

`scanf` reads the input and stores it in **single precision format** in the first half of the **double word** allocated to x, but when x is **printed**, its value is read assuming it is **stored in double precision format**.

When `scanf` reads a **double precision** variable **we must use the format** `%lf` (for long float), so that it stores the result in **double** precision format.

`printf` **expects** double precision, and single precision variables are **automatically** converted to double before being passed to it. Since it always receives **long float** arguments, it treats `%e` and `%le` identically;

likewise `%f` and `%lf`, `%g` and `%lg`.

A program to "test" if x is "zero"

```
main() /* loop1.c: generate small numbers*/
{ float x; int n;
  n = 0; x = 1;   /* x = 2^0 */
  while (x != 0){
    n++;
    x = x/2;    /* x = 2^(-n) */
    printf("\n n= %d  x=%e", n,x); }
}
```

Initializes x to 1 and repeatedly divides by 2 until it rounds to 0. Thus x becomes 1/2, 1/4, 1/8, ..., thru **subnormal** 2^{-127}, ..., 2^{-128}, to **the smallest subnormal** 2^{-149}.

The last value is 0, since 2^{-150} is **not** representable, and rounds to zero.

```
n= 1  x=5.000000e-01
n= 2  x=2.500000e-01
. . .
n= 149  x=1.401298e-45
n= 150  x=0.000000e+00
```

Another "test" if x is "zero"

```
main() /* loop2.c: this loop stops sooner*/
{ float x,y; int n;
  n = 0; y = 2; x = 1;     /* x = 2^0 */
  while (y != 1) {
    n++;
    x = x/2;     /* x = 2^(-n) */
    y = 1 + x;     /* y = 1 + 2^(-n) */
    printf("\n n= %d x= %e y= %e",n,x,y);
  }
}
```

Initializes x to 1 and repeatedly divides by 2, **terminating** when $y = 1 + x$ is 1. This occurs **much sooner**, since $1 + 2^{-24}$ is **not** a floating point number. $1 + 2^{-23}$ does **not** round to 1.

```
n= 1 x= 5.000000e-01 y= 1.500000e+00
 . . .
n= 23 x= 1.192093e-07 y= 1.000000e+00
n= 24 x= 5.960464e-08 y= 1.000000e+00
```

Yet another "test" if x is "zero"

Now instead of using the variable y, change the **test** `while (y != 1)` to `while (1 + x != 1)`.

```
n=0;  x=1;     /* x = 2^0 */
while(1 + x != 1){ ... /* same as before */}
```

The loop now runs until $n = 53$ (`cc` on Sun 4).

```
n= 1 x= 5.000000e-01  y=1.500000e+00
n= 2 x= 2.500000e-01  y=1.250000e+00
. . .
n= 52 x= 2.220446e-16  y=1.000000e+00
n= 53 x= 1.110223e-16  y=1.000000e+00
```

$1 + x$ is computed in a **register** and is **not** stored to memory: the result **in the register** is compared **directly** with the value **1** and, because it uses **double precision**, $1 + x > 1$ for values of n up to 52.

This stops at $n = 64$ on the Pentium both by `cc` and `gcc`, which uses **extended precision** registers.

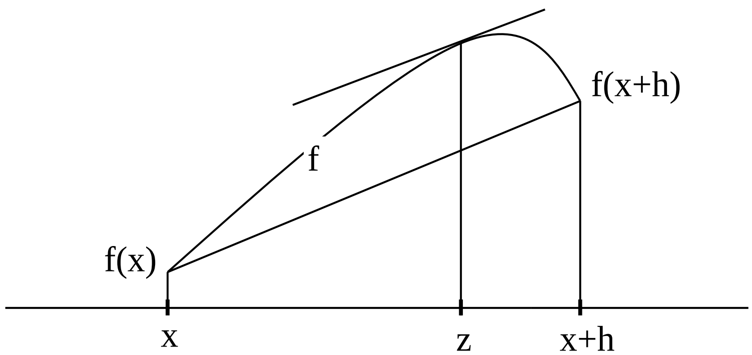
But it still stops at $n = 24$ on the Sun 4 by `gcc`.

The Mean Value Theorem

The **mean value theorem** (**MVT**): Let $f(x)$ be differentiable. For **some** z in $[x, x+h]$:

$$f'(z) = \frac{f(x+h) - f(x)}{h}.$$

This is *intuitively* clear from:



We can **rewrite** the MV formula as:

$$f(x+h) = f(x) + hf'(z).$$

A generalization if f is **twice** differentiable is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(z),$$

for some z in $[x, x+h]$.

Taylor Series

Taylor's Theorem: Let f have continuous derivative of order $0, 1, \ldots, (n+1)$, then

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1},$$

where the error (remainder) $E_{n+1} = \frac{f^{(n+1)}(z)}{(n+1)!} h^{n+1}$, $z \in [x, x+h]$.
Taylor series. If f has **infinitely** many derivatives,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots,$$

where we assume $\lim_{n\to\infty} E_{n+1} = 0$,
e.g. $f(x) = \sin x$:

$$\begin{aligned}\sin(x+h) &= \sin(x) + h\sin'(x) + \frac{h^2}{2!}\sin''(x) \\ &+ \frac{h^3}{3!}\sin'''(x) + \frac{h^4}{4!}\sin''''(x) + \cdots,\end{aligned}$$

Letting $x = 0$, we get

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots.$$

Numerical Approximation to $f'(x)$

If h is **small**, $f'(x)$ is nearly the **slope** of the line through $(x, f(x))$ and $(x+h, f(x+h))$. We write

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

the **forward difference** approximation.
How **good** is this approximation? Using the **truncated Taylor series**:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(z),$$

we get

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{h}{2}f''(z),$$

the **discretization error**: the difference between what we want and our approximation. Using the **discretization size** h. We say the discretization error is $O(h)$.

Computing the Approximation to f'(x)

```
main()  /* diff1.c: approx. deriv. */
{int n; double x,h,approx,exact,error;
 x=1.0; h=1.0; n=0;
 printf("\n h exact approx error");
 while(n<20) {
  n++;
  h=h/10;      /* h=10^(-n) */
  approx=(sin(x+h)-sin(x))/h;/*app.deriv.*/
  exact=cos(x);   /*exact derivative */
  error=approx - exact; /*disczn error */
  printf("\n ... ,h,exact,apprx,err);
 }
}
```

We want to approximate the deriv of $f(x) = \sin(x)$ at $x = 1$. Knowing $\sin'(x) = \cos(x)$, we can compute the **exact** discretization error. The program uses **double precision**, and displays

$$(\sin(x+h) - \sin(x))/h,$$

with the **error**, for h from 0.1 to 10^{-20}.

Convergence of Approximation

```
h       exact         approx        error
e-03 5.403023e-01 5.398815e-01 -4.20825e-04
e-04 5.403023e-01 5.402602e-01 -4.20744e-05
e-05 5.403023e-01 5.402981e-01 -4.20736e-06
e-06 5.403023e-01 5.403019e-01 -4.20746e-07
e-07 5.403023e-01 5.403023e-01 -4.18276e-08
e-08 5.403023e-01 5.403023e-01 -2.96988e-09
e-09 5.403023e-01 5.403024e-01  5.25412e-08
e-10 5.403023e-01 5.403022e-01 -5.84810e-08
e-11 5.403023e-01 5.403011e-01 -1.16870e-06
e-12 5.403023e-01 5.403455e-01  4.32402e-05
e-13 5.403023e-01 5.395684e-01 -7.33915e-04
e-14 5.403023e-01 5.440093e-01  3.70697e-03
e-15 5.403023e-01 5.551115e-01  1.48092e-02
e-16 5.403023e-01 0.000000e+00 -5.40302e-01
```

Discretization error reduced by ~ 10 when h is reduced by 10, so the error **is** $O(h)$. But when h gets **too** small, the approximation starts to get **worse!**
Q: Why **??**

Explanation of Accuracy Loss

If $x = 1$, and $h < \epsilon/2 \approx 10^{-16}$ the **machine epsilon**, $x + h$ has the **same numerical value** as x, and so $f(x + h)$ and $f(x)$ **cancel** to give **0**: the answer has **no digits of precision**.

When h is a **little** bigger than ϵ, the values **partially cancel**. e.g. suppose that the first 10 digits of $f(x+h)$ and $f(x)$ are the same. Then, even though $\sin(x + h)$ and $\sin(x)$ are **accurate to 16 digits**, the **difference** has only **6 accurate digits**

In summary, using h **too big** means a big **discretization** error, while using h **too small** means a big **cancellation** error. For the function $f(x) = \sin(x)$, for example, at $x = 1$, the best choice of h is about 10^{-8}, or $\sim \sqrt{\epsilon}$.

Numerical cancellation, which results when subtraction of **nearly equal values** occurs, should always be avoided when possible.

Numerical Cancellation

One of the main causes for deterioration in precision is numerical cancellation.
Suppose x and y are exact values of two numbers and $x \neq y$. But in many situations we may only have the approximate values

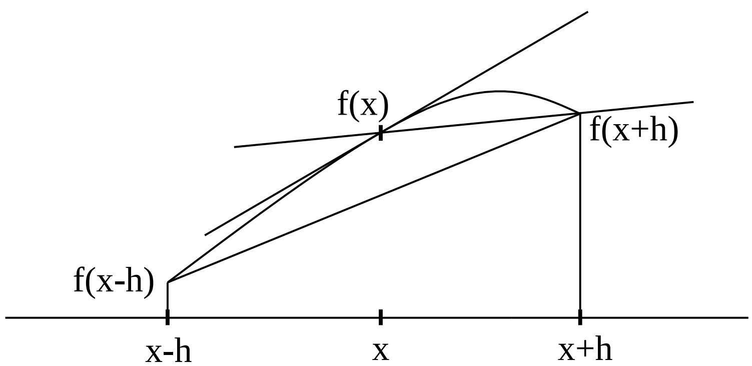
$$\hat{x} = x(1 + \delta_x), \qquad \hat{y} = y(1 + \delta_y),$$

where δ_x and δ_y are the ralative errors in $\hat{x}$ and $\hat{y}$, respectively. These values may be obtained from some computations or physical experiments. Suppose we want to compute $x - y$. But we can only compute $\hat{x} - \hat{y}$. Is it a good approximation of $x - y$?

$$\left|\frac{(\hat{x} - \hat{y}) - (x - y)}{x - y}\right| = \left|\frac{x\delta_x - y\delta_y}{x - y}\right| \leq \frac{|x|}{|x - y|}|\delta_x| + \frac{|y|}{|x - y|}|\delta_y| \leq \max\{|\delta_x|, |\delta_y|\}\frac{|x| + |y|}{|x - y|}$$

This suggests when $|x - y| \ll |x| + |y|$, it is possible that the relative error may be very large.

More accurate numerical differentiation



As h **decreases**, the line through
$(x - h, f(x - h))$ and $(x + h, f(x + h))$ gives a **better approximation** to the **slope of the tangent** to f at x than the line through $(x, f(x))$ and $(x + h, f(x + h))$.
This observation leads to the approximation:

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h},$$

the **central difference** formula.

Analyzing Central Difference Formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Its superiority can be justified as follows.
From the **truncated Taylor series:**

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(z_1),$$

(with z_1 between x and $x+h$). But also

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(z_2),$$

(with z_2 between x and $x-h$).
Subtracting the 2nd from the 1st:

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{12}(f'''(z_1) + f'''(z_2))$$

Discretization error

$$\frac{h^2}{12}(f'''(z_1) + f'''(z_2))$$

is $O(h^2)$ instead of $O(h)$.