An Efficient Algorithm for GPS Based Positioning

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Outline

- Introduction to GPS
- The Mathematical Model
- A Recursive Least Squares Method
- Experimental Results

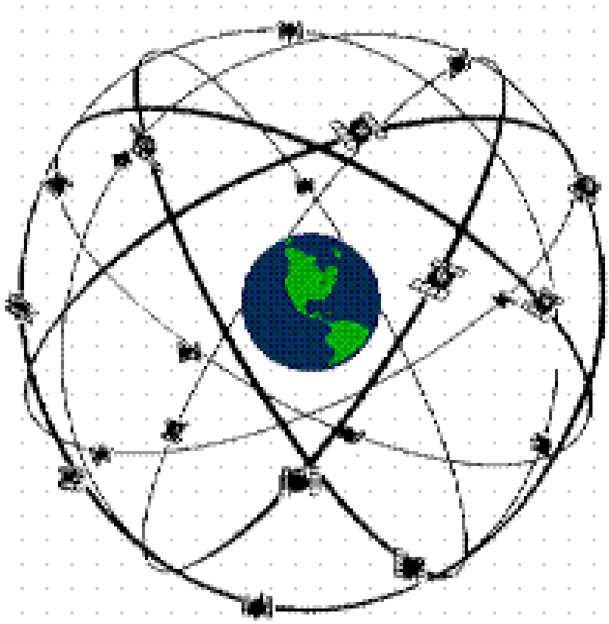
Introduction to GPS

What Is the Global Positioning System (GPS) ?

An all-weather, worldwide, continuous coverage, satellite based navigation system, operated by the U. S. military.

GPS Segments:

- Space: 24 Satellites, in 6 orbits at 20,200 km altitude.
- Control: 1 master control station, 6 monitor stations, 4 ground antennas.
- User: receivers and the users.



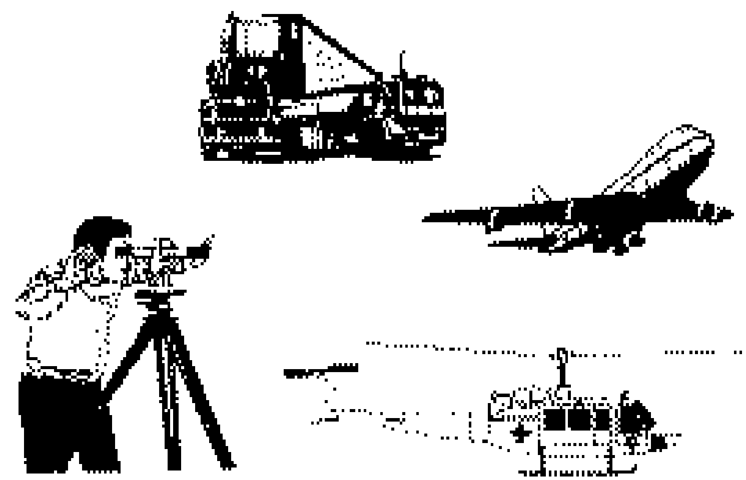
GPS Applications

GPS Military Uses:

GPS has become important for nearly all military operations and weapons systems.

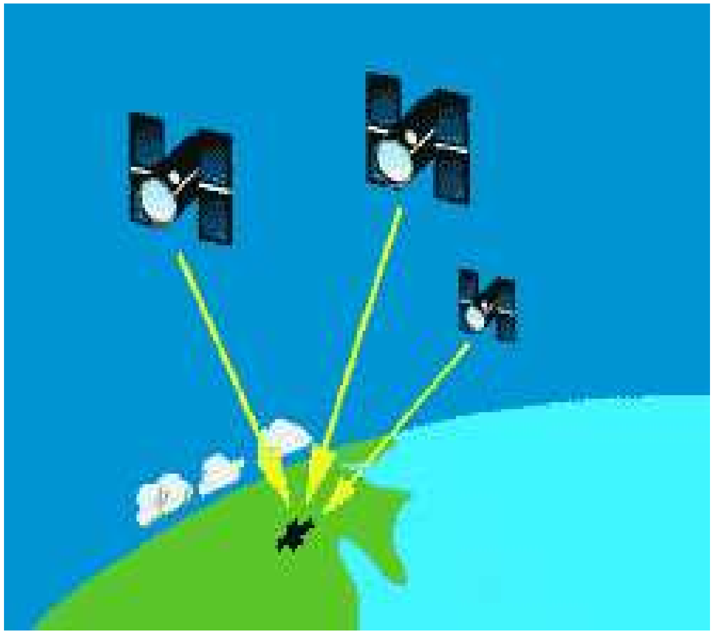
GPS Civil Uses:

- Land/sea/air/space navigation
- Mapping / GIS
- Surveying
- Search and rescue
- Recreation
- Intelligent vehicle highway systems, ...



How GPS Works

- The basis of GPS is "trilateration".
- In principle, if the distances from 3 satellites to a receiver can be measured, the receiver position can be determined.



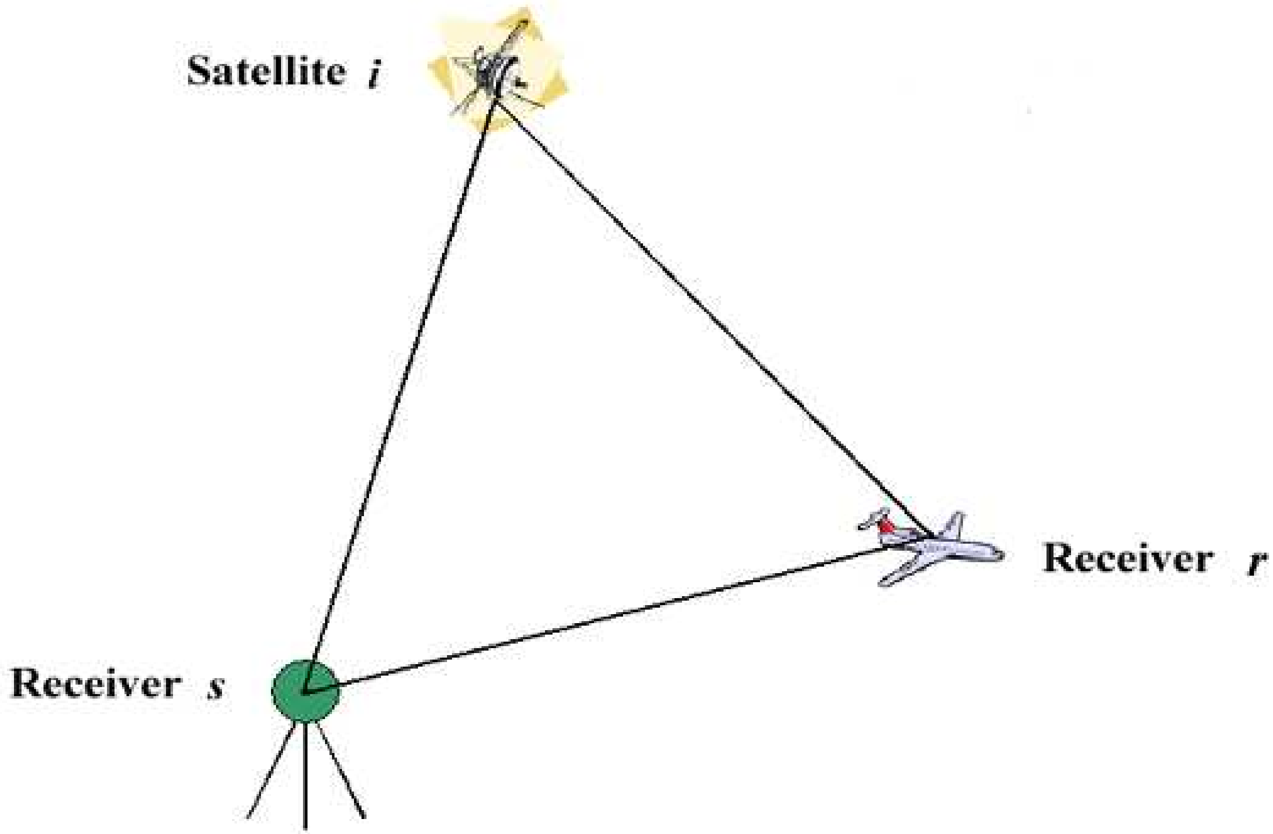
GPS Signal Measurements:

- Code measurements.
- Carrier phase measurements.

Carrier phase measurements are much more complicated but much more accurate than code measurements.

Physical Setting for Differential GPS

Two receivers are close enough, usually within 30 km, such that the received signals have almost the **same** atmospheric errors.



The Mathematical Model (1/2)

Suppose the two receivers obtain the signals simultaneously at t_k.
we have the **single difference** carrier phase meas eqn.

$$\phi_k^i = \lambda^{-1}(\boldsymbol{e}_k^i)^T \boldsymbol{x}_k + \alpha^i + \beta_k + \nu_k^i, \quad \text{(for satellite } i\text{, at time } t_k\text{)}$$

ϕ_k^i: measurement; λ: wavelength; ν_k^i: noise;

$\boldsymbol{x}_k$: baseline vector from receivers s to r; (unknown)

$\boldsymbol{e}_k^i$: unit vector from the midpoint of $\boldsymbol{x}_k$ to satellite i; (unknown)

α^i: integer ambiguity; β_k: receiver clock error. (unknown)

Suppose there are m visible satellites. Then we have m such eqs:

$$\boldsymbol{y}_k = E_k \boldsymbol{x}_k + \boldsymbol{a} + \boldsymbol{e}\,\beta_k + \boldsymbol{v}_k,$$

Usually we assume the noise vectors $\boldsymbol{v}_k$ $(k = 1, 2 \ldots)$ satisfy

$$\boldsymbol{v}_k \sim \mathcal{N}(\mathbf{0}, \sigma^2 I_m), \qquad \mathrm{cov}\{\boldsymbol{v}_k, \boldsymbol{v}_l\} = \mathbf{0}, \quad k \neq l.$$

The Mathematical Model (2/2)

$$\boldsymbol{y}_k = E_k\,\boldsymbol{x}_k + \boldsymbol{a} + \boldsymbol{e}\,\beta_k + \boldsymbol{v}_k$$

For $k = 1, 2, \ldots$, we have **Model M1**:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \cdot \\ \boldsymbol{y}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{e} & E_1 & & & & & I_m \\ & & \boldsymbol{e} & E_2 & & & I_m \\ & & & & \cdot & & \cdot \\ & & & & \boldsymbol{e} & E_k & I_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \boldsymbol{x}_1 \\ \beta_2 \\ \boldsymbol{x}_2 \\ \cdot \\ \beta_k \\ \boldsymbol{x}_k \\ \boldsymbol{a} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \cdot \\ \boldsymbol{v}_k \end{bmatrix} \qquad (M1)$$

Problem: Estimate positions based on Model M1.

Note: $\boldsymbol{a}$ is fixed, & a vector of integers. E_k depends on $\boldsymbol{x}_k$. But given an approximation to $\boldsymbol{x}_k$ (our estimate of $\boldsymbol{x}_{k-1}$ say), we can compute an approximation to E_k.

A Recursive LS Method

Handling Rank Deficiency of Model M1

Let P be a Householder matrix: $P^T \boldsymbol{e} = \sqrt{m}\,[1, 0, ..., 0]^T$.

Apply $P^T \equiv [\underset{1}{\boldsymbol{p}_1}, \underset{m-1}{P_2}]^T$ to $\boldsymbol{y}_j = E_j \boldsymbol{x}_j + \boldsymbol{a} + \boldsymbol{e}\beta_j + \boldsymbol{v}_j$:

$$
\begin{aligned}
\boldsymbol{p}_1^T \boldsymbol{y}_j &= \boldsymbol{p}_1^T E_j \boldsymbol{x}_j + \underline{\boldsymbol{p}_1^T \boldsymbol{a} + \sqrt{m}\,\beta_j} + \boldsymbol{p}_1^T \boldsymbol{v}_j \\
&\equiv \boldsymbol{p}_1^T E_j \boldsymbol{x}_j + \underline{\gamma_j} + \boldsymbol{p}_1^T \boldsymbol{v}_j \\
P_2^T \boldsymbol{y}_j &= P_2^T E_j \boldsymbol{x}_j + \underline{P_2^T \boldsymbol{a}} + P_2^T \boldsymbol{v}_j \\
&\equiv P_2^T E_j \boldsymbol{x}_j + \underline{\boldsymbol{d}} + P_2^T \boldsymbol{v}_j
\end{aligned}
$$

Note: One variable has been eliminated.

$P^T \boldsymbol{v}_j \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 I_m)$.

A Recursive LS Method

$$\boldsymbol{p}_1^T \boldsymbol{y}_j = \boldsymbol{p}_1^T E_j \boldsymbol{x}_j + \gamma_j + \boldsymbol{p}_1^T \boldsymbol{v}_j,$$
$$P_2^T \boldsymbol{y}_j = P_2^T E_j \boldsymbol{x}_j + \boldsymbol{d} + P_2^T \boldsymbol{v}_j.$$

Combining these for $j = 1, \ldots, k$ gives

$$\left[\begin{array}{c} \boldsymbol{p}_1^T \boldsymbol{y}_1 \\ \cdot \\ \boldsymbol{p}_1^T \boldsymbol{y}_k \\ \hline P_2^T \boldsymbol{y}_1 \\ \cdot \\ P_2^T \boldsymbol{y}_k \end{array}\right] = \left[\begin{array}{ccc|ccc|c} 1 & & & \boldsymbol{p}_1^T E_1 & & & \\ & \cdot & & & \cdot & & \\ & & 1 & & & \boldsymbol{p}_1^T E_k & \\ \hline & & & P_2^T E_1 & & & I_{m-1} \\ & & & & \cdot & & \cdot \\ & & & & & P_2^T E_k & I_{m-1} \end{array}\right] \left[\begin{array}{c} \gamma_1 \\ \cdot \\ \gamma_k \\ \hline \boldsymbol{x}_1 \\ \cdot \\ \boldsymbol{x}_k \\ \hline \boldsymbol{d} \end{array}\right] + \left[\begin{array}{c} \boldsymbol{p}_1^T \boldsymbol{v}_1 \\ \cdot \\ \boldsymbol{p}_1^T \boldsymbol{v}_k \\ \hline P_2^T \boldsymbol{v}_1 \\ \cdot \\ P_2^T \boldsymbol{v}_k \end{array}\right]$$

Estimate $\boldsymbol{d}$ & $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$; then $\gamma_1, \ldots, \gamma_k$ follow.

A Recursive LS Method (3/7)

Problem: Estimate positions based on Model M2:

$$\begin{bmatrix} P_2^T \boldsymbol{y}_1 \\ \cdot \\ P_2^T \boldsymbol{y}_k \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} P_2^T E_1 & & & I_{m-1} \\ & \cdot & & \cdot \\ & & P_2^T E_k & I_{m-1} \end{array}\right]}_{k(m-1)\times(3k+m-1)} \underbrace{\left[\begin{array}{c} \boldsymbol{x}_1 \\ \cdot \\ \boldsymbol{x}_k \\ \hline \boldsymbol{d} \end{array}\right]}_{\text{unknowns}} + \underbrace{\begin{bmatrix} P_2^T \boldsymbol{v}_1 \\ \cdot \\ P_2^T \boldsymbol{v}_k \end{bmatrix}}_{\text{noise}} \qquad (M2)$$

$$P_2^T \boldsymbol{y}_j = P_2^T E_j \boldsymbol{x}_j + \boldsymbol{d} + P_2^T \boldsymbol{v}_j$$

A Recursive LS Method (4/7)

When does Model M2 have full column rank (FCR) ?

The matrix in Model M2 is $k(m-1) \times (3k+m-1)$.
In order to have FCR, we require

$$k(m-1) \geq 3k+m-1.$$

Requirement of number of epochs and satellites

# of epochs (k)	2	3	4	5	≥ 6
minimum # of satellites (m)	7	6	5	5	5

Can show if the column spaces of $P_2^T E_1, \ldots, P_2^T E_k$ do not intersect, Model M2 has FCR.

A Recursive LS Method (5/7)

Computing the LS Solution of Model M2

$$P_2^T \boldsymbol{y}_j = P_2^T E_j \boldsymbol{x}_j + \boldsymbol{d} + P_2^T \boldsymbol{v}_j, \qquad j = 1, \ldots, k$$

Let the QR factn of $P_2^T E_j$ be

$$Q_j^T (P_2^T E_j) = \begin{bmatrix} R_j \\ \mathbf{0} \end{bmatrix}, \quad Q_j^T \equiv \begin{bmatrix} U_j \\ W_j \end{bmatrix}, \quad Q_j^T (P_2^T \boldsymbol{y}_j) = \begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{w}_j \end{bmatrix},$$

where Q_j is orthogonal, R_j is upper triangular. This gives

$$\begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix} = \begin{bmatrix} R_j \\ \mathbf{0} \end{bmatrix} \boldsymbol{x}_j + \begin{bmatrix} U_j \\ W_j \end{bmatrix} \boldsymbol{d} + \begin{bmatrix} U_j P_2^T \boldsymbol{v}_j \\ W_j P_2^T \boldsymbol{v}_j \end{bmatrix}.$$

Computing the LS Solution of Model M1, contd

$$\begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix} = \begin{bmatrix} R_j \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x}_j + \begin{bmatrix} U_j \\ W_j \end{bmatrix} \boldsymbol{d} + \begin{bmatrix} U_j P_2^T \boldsymbol{v}_j \\ W_j P_2^T \boldsymbol{v}_j \end{bmatrix}.$$

Transforming Model M2 by $\mathrm{diag}(Q_1^T, \ldots, Q_k^T)$ and reordering gives

$$\left[\begin{array}{c} \boldsymbol{u}_1 \\ \cdot \\ \boldsymbol{u}_k \\ \hline \boldsymbol{w}_1 \\ \cdot \\ \boldsymbol{w}_k \end{array}\right] = \left[\begin{array}{ccc|c} R_1 & & & U_1 \\ & \cdot & & \cdot \\ & & R_k & U_k \\ \hline & & & W_1 \\ & & & \cdot \\ & & & W_k \end{array}\right] \underbrace{\left[\begin{array}{c} \boldsymbol{x}_1 \\ \cdot \\ \boldsymbol{x}_k \\ \hline \boldsymbol{d} \end{array}\right]}_{\text{unknown}} + \underbrace{\left[\begin{array}{c} U_1 P_2^T \boldsymbol{v}_1 \\ \cdot \\ U_k P_2^T \boldsymbol{v}_k \\ \hline W_1 P_2^T \boldsymbol{v}_1 \\ \cdot \\ W_k P_2^T \boldsymbol{v}_k \end{array}\right]}_{\text{noise}}.$$

Estimate $\boldsymbol{d}$, solve for any of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

A Recursive LS Method (7/7)

Compute the LS estimate d_k for d recursively

— by orthogonal transformations:

$$\boldsymbol{d}_k = \arg\min_{\boldsymbol{d}} \left\| \begin{bmatrix} \boldsymbol{w}_1 \\ \cdot \\ \boldsymbol{w}_k \end{bmatrix} - \begin{bmatrix} W_1 \\ \cdot \\ W_k \end{bmatrix} \boldsymbol{d} \right\|_2$$

Compute the LS estimates $\boldsymbol{x}_{j|k}$ for $\boldsymbol{x}_j$ at epoch k:

$$\begin{bmatrix} \boldsymbol{u}_1 \\ \cdot \\ \boldsymbol{u}_k \end{bmatrix} = \left[\begin{array}{ccc|c} R_1 & & & U_1 \\ & \cdot & & \cdot \\ & & R_k & U_k \end{array}\right] \begin{bmatrix} \boldsymbol{x}_1 \\ \cdot \\ \boldsymbol{x}_k \\ \hline \boldsymbol{d} \end{bmatrix}$$

$\Longrightarrow R_j \boldsymbol{x}_{j|k} = \boldsymbol{u}_j - U_j \boldsymbol{d}_k, \quad j = 1, \ldots, k, \quad (R_j$ upper triangular).

Remarks

- **Initial stage**:

  For the initial epochs, the estimation problem does not have FCR, we use another technique to estimate the positions until the problem becomes FCR.

- **Integer ambiguities**:

  If we want to obtain centimeter accuracy in as few epochs as possible, the integer nature of the ambiguities should be used. Our algorithm allows us to fix the integer ambiguities.

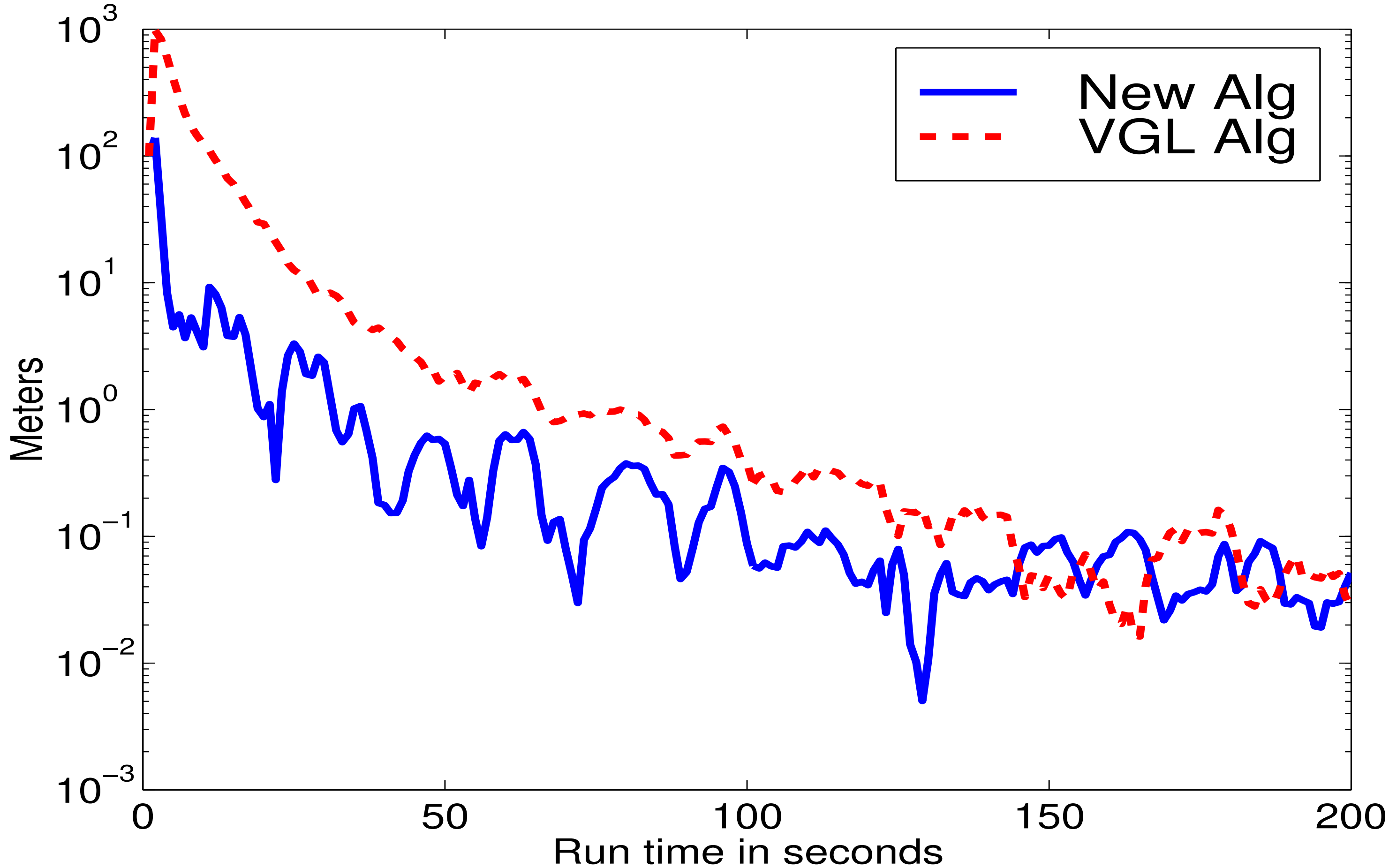
- **Number of satellites**:

  We assumes we always had the same visible satellites. But it is not difficult to modify the algorithm to handle satellite rising and setting.

Experimental Results

Computer simulations:

- A 24 GPS satellite constellation
  —The data was obtained from the Internet.
- The rover is on board an aircraft circling above a reference station at $100\,m/s$.
- Simulate the van Graas and Lee alg ('95), which uses a "complementary Kalman filtering" technique.



Position errors versus run time, with 100 m initial error.

1 km baseline, 7 satellites, noise $\sigma = 0.2$ cm.

At each epoch,

$$\begin{aligned} & \text{number of flops of the new alg} \\ \leq\ & \frac{1}{2}\,\text{number of flops of the VGL alg} \end{aligned}$$

Summary

- Basic ideas of GPS were introduced.
- Carrier phase based recursive LS approach for positioning.
- The algorithm is efficient—makes full use of structure.
- & numerically reliable—uses orthogonal transformations.