Mechanizing Meta-Theory in Beluga

Brigitte Pientka
School of Computer Science
McGill University
Montreal, Canada

Joint work with Andrew Cave
Introduction

Beluga: Design and implementation

How to mechanize formal systems and proofs?

- Formal systems (given via axioms and inference rules) play an important role when designing languages and more generally software.

- Proofs (that a given property is satisfied) are an integral part of the software (see: certified code, proof-carrying architectures).
How to mechanize formal systems and proofs?

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- Proofs (that a given property is satisfied) are an integral part of the software (see: certified code, proof-carrying architectures).

**Properties**

- Memory Safety: Program does not crash
- Authenticity: Communicates only within domain mcgill.ca
- Type Safety: Execution of program does not go wrong
Question

What are good meta-languages to program and reason with formal systems and proofs?
“We may think of [the] proof as an iceberg. In the top of it, we find what we usually consider the real proof; underwater, the most of the matter, consisting of all mathematical preliminaries a reader must know in order to understand what is going on.”

S. Berardi [1990]
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S. Berardi [1990]
“The motivation behind the work in very-high-level languages is to ease the programming task by providing the programmer with a language containing primitives or abstractions suitable to his problem area. The programmer is then able to spend his effort in the right place; he concentrates on solving his problem, and the resulting program will be more reliable as a result. Clearly, this is a worthwhile goal.”  

B. Liskov [1974]
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B. Liskov [1974]
Above and Below the Surface

**Beluga:** Dependently typed Programming and Proof Environment

- **Below the surface:** Support for key concepts based on Contextual LF
- **Above the surface:** Proofs by structural Induction = Recursive Programs
  
  First-order Logic over Contextual LF objects (i.e. Contexts, Derivation trees, Substitutions, ...) together with inductive definitions and induction principles
This Talk

Design and implementation of Beluga

- Introduction
- Example: Proof by logical relation
- Writing a proof in Beluga . . .
- Conclusion and current work

“The limits of my language mean the limits of my world.”
- L. Wittgenstein
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Simply Typed Lambda-calculus (Gentzen-style)

Types $A, B ::= \ i$ $|\ A \Rightarrow B$

Terms $M, N ::= x | c$ $|\ \text{lam } x.M$ $|\ \text{app } M N$

Evaluation Judgment: $M \rightarrow M'$

- $\text{app (lam } x.M) N \rightarrow [N/x]M$ s/beta
- $M \rightarrow M'$ s/app
- $M \rightarrow M'$ $M \rightarrow N$ s/trans

read as “$M$ steps to $M’$”

Typing Judgment: $M : A$ read as “$M$ has type $A$” (Gentzen-style)

- $c : i$ const
- $x : A \ u$
- $\text{lam } x.M : A \Rightarrow B$
- $\text{app } M N : B$
- $M : A \Rightarrow B N : A$ s/refl
Simply Typed Lambda-calculus (Gentzen-style)

Types $A, B ::= i$

$\mid A \Rightarrow B$

Terms $M, N ::= x \mid c$

$\mid \text{lam } x. M$

$\mid \text{app } M N$

Evaluation Judgment: $M \rightarrow M'$

read as “$M$ steps to $M$”

$\frac{\text{app } (\text{lam } x. M) N \rightarrow [N/x]M}{s/\text{beta}}$

$\frac{M \rightarrow M'}{s/\text{app}}$

$\frac{M \rightarrow M'}{s/\text{refl}}$

$\frac{M \rightarrow M' \quad M' \rightarrow N}{s/\text{trans}}$

Typing Judgment: $M : A$

read as “$M$ has type $A$” (Gentzen-style)

$\frac{x : A \quad \vdots}{\text{lam } x. M : A \Rightarrow B}$

$\frac{c : i \quad \text{const}}{\text{app } M N : B}$

\[\text{lam}^{x,u} \quad \text{app} \quad \text{const} \]
Simply Typed Lambda-calculus with Contexts

Types and Terms

Types \( A, B \ ::= \ i \) 

| \( A \Rightarrow B \) 

Terms \( M, N \ ::= \ x \) 

| \( c \) 

| \( \text{lam} \ x. \ M \) 

| \( \text{app} \ M \ N \)

Evaluation Judgment: \( \boxed{M \rightarrow M'} \) read as “\( M \) steps to \( M' \)”

\[
\frac{\text{app} (\text{lam} x. M) \ N \rightarrow [N/x]M}{s/\text{beta}}
\]

\[
\frac{M \rightarrow M'}{s/\text{app}}
\]

\[
\frac{\text{app} M \ N \rightarrow \text{app} M' \ N}{s/\text{trans}}
\]

Typing Judgment: \( \Gamma \vdash M : A \) read as “\( M \) has type \( A \) in context \( \Gamma \)”

\[
\frac{x : A \in \Gamma}{\Gamma \vdash x : A}
\]

\[
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \text{lam} x. M : A \Rightarrow B}
\]

\[
\frac{\Gamma \vdash M : A \Rightarrow B \ \Gamma \vdash N : A}{\Gamma \vdash \text{app} M \ N : B}
\]

Context \( \Gamma \ ::= \cdot | \Gamma, x : A \) We are introducing the variable \( x \) together with the assumption \( x : A \)
Derivations Under the Magnifying Glass

Typing rules

\[
\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \text{lam} \, x \, . \, M : A \Rightarrow B} \quad \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : B}{\Gamma \vdash \text{app} \, M \, N : B}
\]

Evaluation rules

\[
\frac{\text{app} \, (\text{lam} \, x \, . \, M) \, N \rightarrow [N/x]M}{M \rightarrow M'} \quad \frac{\text{app} \, M \, N \rightarrow \text{app} \, M' \, N}{M \rightarrow M'}
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Derivations Under the Magnifying Glass

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\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \Rightarrow B}
\]

Evaluation rules

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\frac{\text{app} \ (\lambda x. M) \ N \rightarrow [N/x]M}{\text{beta}} \quad \frac{M \rightarrow M'}{\text{app}} \quad \frac{\text{app} \ M \ N \rightarrow \text{app} \ M' \ N}{\text{app}}
\]

• What kinds of variables are used?

Bound variables, Eigenvariables, Schematic variables, Context variables

• What operations on variables are needed?

Substitution and Renaming for bound variable, Substitution for schematic variables, Substitution for hypothesis and eigenvariables

• How should we represent contexts? What properties do contexts have?

(Structured) sequences, Uniqueness of declaration, Weakening, Substitution lemma, etc.

Any mechanization of proofs must deal with these issues.
Derivations Under the Magnifying Glass

Typing rules

\[
\begin{align*}
\frac{x : A \in \Gamma}{\Gamma \vdash x : A} & \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \text{lam} \, x. \, M : A \Rightarrow B} \\
\end{align*}
\]

Evaluation rules

\[
\begin{align*}
\frac{\text{app} \, (\text{lam} \, x. \, M) \, N \rightarrow [N/x]M}{\text{beta}} & \quad \frac{M \rightarrow M'}{\text{app} \, M \, N \rightarrow \text{app} \, M' \, N} \\
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Weak Normalization for Simply Typed Lambda-calculus

Theorem

If $\vdash M : A$ then $M$ halts, i.e. there exists a value $V$ s.t. $M \rightarrow^* V$.

Proof.

1 Define reducibility candidate $R_A$:

$R_A \triangleright B = \{ M | M \text{ halts and } \forall N \in R_A, (\text{app } M N) \in R_B \}$

2 If $M \in R_A$ then $M$ halts.

3 Backwards closed: If $M' \in R_A$ and $M \rightarrow M'$ then $M \in R_A$.

4 Fundamental Lemma: If $\vdash M : A$ then $M \in R_A$. (Requires a generalization)

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Weak Normalization for Simply Typed Lambda-calculus

Theorem

If ⊢ M : A then M halts, i.e. there exists a value V s.t. M →* V.

Proof.

1. Define reducibility candidate \( R_A \)

\[
R_i = \{ M \mid M \text{ halts} \}
\]
\[
R_{A⇒B} = \{ M \mid M \text{ halts and } \forall N \in R_A, (\text{app } M N) \in R_B \}
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2. If \( M \in R_A \) then \( M \) halts.

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4. Fundamental Lemma: If \( ⊢ M : A \) then \( M \in R_A \). (Requires a generalization)
Generalization of Fundamental Lemma

Lemma (Main lemma)

If \( D: \Gamma \vdash M : A \) and \( \sigma \in \mathcal{R}_\Gamma \) then \([\sigma]M \in \mathcal{R}_A\).

where \( \sigma \in \mathcal{R}_\Gamma \) is defined as:

\[
\begin{align*}
\cdot & \in \mathcal{R}.
\sigma & \in \mathcal{R}_\Gamma \quad N \in \mathcal{R}_A \\
(\sigma, N/x) & \in \mathcal{R}_{\Gamma,x:A}
\end{align*}
\]
**Generalization of Fundamental Lemma**

**Lemma (Main lemma)**

*If* $\mathcal{D} : \Gamma \vdash M : A$ *and* $\sigma \in R_\Gamma$ *then* $[\sigma]M \in R_A$.

**Proof.**

**Case** $\mathcal{D} = \Gamma, x : A \vdash M : B$

$$\Gamma \vdash \text{lam } x.M : A \Rightarrow B \quad \text{lam}$$

$[\sigma](\text{lam } x.M) = \text{lam } x.([\sigma, x/x]M)$

halts $(\text{lam } x. [\sigma, x/x]M)$

Suppose $N \in R_A$.

$[\sigma, N/x]M \in R_B$

$[N/x][\sigma, x/x]M \in R_B$

app $(\text{lam } x. [\sigma, x/x]M) N \in R_B$

Hence $[\sigma](\text{lam } x.M) \in R_{A \Rightarrow B}$
Challenging Benchmark

“I discovered that the core part of the proof (here proving lemmas about CR) is fairly straightforward and only requires a good understanding of the paper version. However, in completing the proof I observed that in certain places I had to invest much more work than expected, e.g. proving lemmas about substitution and weakening.”

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T. Altenkirch [TLCA’93]

- Binders: lambda-binder, $\forall$ in reducibility definition, quantification over substitutions and contexts
- Contexts: Uniqueness of assumptions, weakening, etc.
- Simultaneous substitution and algebraic properties:
  Substitution lemma, composition, decomposition, associativity, identity, etc.

$$[\cdot]M = M$$
$$[\sigma, N/x]M = [N/x][\sigma, x/x]M$$
$$[\sigma_1][\sigma_2]M = [[[\sigma_1]\sigma_2]M$$

a dozen such properties are needed
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Beluga\(\mu\): Two Level Approach

The Top: Functional programming with indexed types [POPL’08, POPL’12]

Proof term language for first-order logic over a specific domain (= contextual LF) together inductive definitions (= relations) about domain objects and domain-specific induction principle [TLCA’15]
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The Bottom: Contextual logical framework LF [HHP’93,TOCL’08]

- Compact representation of formal systems and derivations
- Higher-order abstract syntax trees and dependent types
  - \(\sim\) support for \(\alpha\)-renaming, substitution, adequate representations
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- Compact representation of formal systems and derivations
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  \(\leadsto\) support for \(\alpha\)-renaming, substitution, adequate representations
- Contextual LF: Contextual types characterize contextual objects [TOCL’08]
  \(\leadsto\) support well-scoped derivations
  \(\leadsto\) abstract notion of contexts and substitution [POPL’08,LFMTP’13]
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  \(\leadsto\) support well-scoped derivations
  \(\leadsto\) abstract notion of contexts and substitution [POPL’08, LFMTP’13]
Step 1: Represent Types and Lambda-terms in LF

Types $A, B ::= i | A \Rightarrow B$

Terms $M, N ::= x | c | \text{lam } x.M | \text{app } M N$

Typing rules

\[
\frac{x : A \quad u}{\text{const}} \quad \frac{M : B}{\text{lam } x.M : A \Rightarrow B} \quad \frac{M : A \Rightarrow B \quad N : A}{\text{app } M N : B}
\]

LF representation in Beluga

\[
\begin{align*}
\text{LF } &\text{tp: type } = \\
&| i : \text{tp} \\
&| \text{arr: tp } \rightarrow \text{tp } \rightarrow \text{tp};
\end{align*}
\]

\[
\begin{align*}
\text{LF } &\text{tm: type } = \\
&| c : \text{tm i} \\
&| \text{lam: (tm A } \rightarrow \text{tm B) } \rightarrow \text{tm (arr A B)} \\
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Step 1: Represent Types and Lambda-terms in LF

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\end{array}
\]

LF representation in Beluga

\[
\begin{array}{l}
\text{LF } \text{tp}: \text{type } = \\
\quad | \ i : \text{tp} \\
\quad | \ \text{arr} : \text{tp } \to \text{tp } \to \text{tp};
\end{array}
\]

\[
\begin{array}{l}
\text{LF } \text{tm}: \text{tp } \to \text{type } = \\
\quad | \ c : \text{tm } i \\
\quad | \ \text{lam} : (\text{tm } A \to \text{tm } B) \to \text{tm } (\text{arr } A B) \\
\quad | \ \text{app} : \text{tm } (\text{arr } A B) \to \text{tm } A \to \text{tm } B;
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\]
Reducibility Candidates as Indexed Types

Reducibility candidates for terms $M \in \mathcal{R}_A$:

\[
\mathcal{R}_i = \{ M \mid \text{halts } M \}
\]

\[
\mathcal{R}_{A \Rightarrow B} = \{ M \mid \text{halts } M \text{ and } \forall N \in \mathcal{R}_A, (\text{app } M N) \in \mathcal{R}_B \}
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Computation-level data types in Beluga

stratified \text{Reduce} : \{A: [\vdash \text{tp}]\} \{M: [\vdash \text{tm } A]\} \text{ type } =
| I : [\vdash \text{halts } M] \rightarrow \text{Reduce } [\vdash i] [\vdash M] |
| Arr : [\vdash \text{halts } M] \rightarrow 
\quad (\{N: [\vdash \text{tm } A]\} \text{Reduce } [\vdash A] [\vdash N] \rightarrow \text{Reduce } [\vdash B] [\vdash \text{app } M N])
\quad \rightarrow \text{Reduce } [\vdash \text{arr } A B] [\vdash M];

- \([\vdash \text{app } M N]\) and \([\vdash \text{arr } A B]\) are contextual types [TOCL'08].
- \(\rightarrow\) is overloaded.
  - \(\rightarrow\) is the LF function space: binders in the object language are modelled by LF functions (used inside \([ ]\))
  - \(\rightarrow\) is a computation-level function (used outside \([ ]\))
- Not strictly positive definition, but stratified.
Reducibility Candidates as Indexed Types

Reducibility candidates for substitutions $\sigma \in \mathcal{R}_\Gamma :$

$\cdot \in \mathcal{R}. \quad \frac{\sigma \in \mathcal{R}_\Gamma \quad N \in \mathcal{R}_A}{(\sigma, N/x) \in \mathcal{R}_{\Gamma,x:A}}$
# Reducibility Candidates as Indexed Types

Reducibility candidates for substitutions $\sigma \in R_\Gamma$:

\[
\begin{align*}
\cdot & \in R. \\
\sigma \in R_\Gamma & \quad \quad N \in R_A \\
(\sigma, N/x) & \in R_{\Gamma, x:A}
\end{align*}
\]

## Computation-level data types in Beluga

**inductive** RedSub : (\(\Gamma:ctx\)){}{\(\sigma: \vdash \Gamma\)} type =

| Nil : RedSub [ \(\vdash \top\) ] |
| Cons : RedSub [ \(\vdash \sigma\) ] \(\rightarrow\) Reduce [ \(\vdash A\) ] [ \(\vdash M\) ] \(\rightarrow\) RedSub [ \(\vdash \sigma \ M\) ]; |

- Contexts are structured sequences and are classified by context schemas
  
  **schema** ctx = x:tm A.

- Substitution $\tau$ are first-class and have type $\Psi \vdash \Phi$ providing a mapping from $\Phi$ to $\Psi$. 
**Lemma (Backward closed)**

\[
\text{If } M \rightarrow M' \text{ and } M' \in \mathcal{R}_A \text{ then } M \in \mathcal{R}_A.
\]

**Rec** closed : \( \vdash \text{mstep } M M' \) \( \rightarrow \) \( \text{Reduce } \vdash A \) \( \vdash M' \) \( \rightarrow \) \( \text{Reduce } \vdash A \) \( \vdash M \) = ? ;

**Lemma (Main lemma)**

\[
\text{If } \Gamma \vdash M : A \text{ and } \sigma \in \mathcal{R}_\Gamma \text{ then } [\sigma]M \in \mathcal{R}_A.
\]

**Rec** main : \{\( \Gamma : \text{ctx}\}\{M : [\Gamma \vdash \text{tm } A]\} \text{ RedSub } \vdash \sigma \rightarrow \text{Reduce } \vdash A \) \( \vdash M \sigma \) = ? ;
Fundamental Lemma

\[ \text{rec \  main} : \{ \Gamma : \text{ctx} \} \{ M : \Gamma \vdash \text{tm} \ A \} \ \text{RedSub} \{ \Gamma \vdash A \} \{ M \sigma \} = \ ? ; \]

\[ \text{mlam} \ \Gamma \Rightarrow \text{mlam} \ M \Rightarrow \text{fn} \ \text{rs} \Rightarrow \text{case} \{ \Gamma \vdash M \} \] of

\[ | \{ \Gamma \vdash \# p \} \Rightarrow \text{lookup} \{ \Gamma \} \{ \Gamma \vdash \# p \} \text{rs} \ % \text{Variable} \]

\[ | \{ \Gamma \vdash \text{app} M_1 M_2 \} \Rightarrow % \text{Application} \]

\[ \text{let} \ \\
\text{Arr} \ ha \ f = \text{main} \{ \Gamma \} \{ \Gamma \vdash M_1 \} \text{rs} \] in

\[ f \ \{ \Gamma \vdash _\} \ (\text{main} \{ \Gamma \} \{ \Gamma \vdash M_2 \} \text{rs}) \]

\[ | \{ \Gamma \vdash \text{lam} \lambda x. M_1 \} \Rightarrow % \text{Abstraction} \]

\[ \text{Arr} \ \{ \Gamma \vdash h/value \ s/refl \ v/lam \} (\text{mlam} N \Rightarrow \text{fn} rN \Rightarrow \text{closed} \{ \Gamma \vdash s/beta \} (\text{main} \{ \Gamma, x : \text{tm} _\} \{ \Gamma, x \vdash \text{M}_1 \} (\text{Cons} \ \text{rs} \ rN))) \]
Fundamental Lemma

\[ \text{rec closed : } [\vdash \text{mstep } M M'] \rightarrow \text{Reduce } [\vdash A] [\vdash M'] \rightarrow \text{Reduce } [\vdash A] [\vdash M] = ? ; \]

\[ \text{rec main : } \{\Gamma : \text{ctx}\} \{M : [\Gamma \vdash \text{tm } A]\} \text{ RedSub } [\vdash \sigma] \rightarrow \text{Reduce } [\vdash A] [\vdash M \sigma] = \]
**Fundamental Lemma**

\[
\text{rec closed : } [\vdash \text{mstep } M \ M'] \rightarrow \text{Reduce } [\vdash A] [\vdash M'] \rightarrow \text{Reduce } [\vdash A] [\vdash M] = \, ? \, ;
\]

\[
\text{rec main : } \{\Gamma : \text{ctx}\}\{M : [\Gamma \vdash \text{tm } A]\} \text{ RedSub } [\vdash \sigma] \rightarrow \text{Reduce } [\vdash A] [\vdash M \ \sigma] =
\]

\[
\text{mlam } \Gamma \Rightarrow \text{mlam } M \Rightarrow \text{fn } \text{rs } \Rightarrow \text{case } [\Gamma \vdash M ] \text{ of}
\]

\[
| [\Gamma \vdash \#p ] \Rightarrow \text{lookup } [\Gamma] [\Gamma \vdash \#p ] \text{ rs } \quad \% \text{ Variable}
\]
Introduction Beluga: Design and implementation

Fundamental Lemma

```
rec closed : [ ⊢ mstep M M’] → Reduce [ ⊢ A] [ ⊢ M’] →Reduce [ ⊢ A] [ ⊢ M] = ? ;
rec main : {Γ:ctx}{M:Γ ⊢ tm A} RedSub [ ⊢ σ] →Reduce [ ⊢ A] [ ⊢ M σ] =
                
mlam Γ⇒mlam M ⇒fn rs ⇒ case [Γ ⊢ M] of
                % Variable
  | [Γ ⊢ #p ] ⇒ lookup [Γ] [Γ ⊢ #p ] rs

  | [Γ ⊢ app M1 M2] ⇒
  let Arr ha f = main [Γ] [Γ ⊢ M1] rs in
  f [ ⊢ _ ] (main [Γ] [Γ ⊢ M2] rs)

| [Γ ⊢ c] ⇒ I [Γ ⊢ h/value s/refl v/c];  % Constant
```
Fundamental Lemma

\[
\text{rec } \text{closed} : \left[ \vdash \text{mstep } M \; M' \right] \to \text{Reduce } \left[ \vdash A \right] \left[ \vdash M' \right] \to \text{Reduce } \left[ \vdash A \right] \left[ \vdash M \right] = \?
\]

\[
\text{rec } \text{main} : \left\{ \Gamma : \text{ctx} \right\} \left\{ M : \left[ \Gamma \vdash \text{tm } A \right] \right\} \text{RedSub } \left[ \vdash \sigma \right] \to \text{Reduce } \left[ \vdash A \right] \left[ \vdash M \sigma \right] =
\]

\[
\text{mlam } \Gamma \Rightarrow \text{mlam } M \Rightarrow \text{fn } \text{rs} \Rightarrow \text{case } \left[ \Gamma \vdash M \right] \text{ of }
\]

| \left[ \Gamma \vdash \#p \right] \Rightarrow \text{lookup } \left[ \Gamma \right] \left[ \Gamma \vdash \#p \right] \; \text{rs} & \%	ext{ Variable} \\
| \left[ \Gamma \vdash \text{app } M_1 \; M_2 \right] \Rightarrow & \%	ext{ Application} \\
\quad \text{let } \text{Arr } h/a = \text{main } \left[ \Gamma \right] \left[ \Gamma \vdash M_1 \right] \; \text{rs in} \\
\quad f \left[ \vdash _\_ \right] \left( \text{main } \left[ \Gamma \right] \left[ \Gamma \vdash M_2 \right] \; \text{rs} \right) & \\
| \left[ \Gamma \vdash \text{lam } \lambda x. \; M_1 \right] \Rightarrow & \%	ext{ Abstraction} \\
\quad \text{Arr } \left[ \vdash h/\text{value } s/\text{refl } v/\text{lam} \right] \\
\quad \left( \text{mlam } N \Rightarrow \text{fn } rN \Rightarrow \text{closed } \left[ \vdash s/\text{beta} \right] \left( \text{main } \left[ \Gamma, x : \text{tm } _\_ \right] \left[ \Gamma, x \vdash M_1 \right] \left( \text{Cons } \text{rs} \; rN \right) \right) \right)
\]
Fundamental Lemma

```
rec closed : [ ⊢ mstep M M'] → Reduce [ ⊢ A] [ ⊢ M'] →Reduce [ ⊢ A] [ ⊢ M] = ? ;
rec main : {Γ:ctx}{M:[Γ ⊢ tm A]} RedSub [ ⊢ σ] →Reduce [ ⊢ A] [ ⊢ M σ] =
mlam Γ⇒mlam M ⇒ fn rs ⇒ case [Γ ⊢ M ] of
| [Γ ⊢ #p ] ⇒ lookup [Γ] [Γ ⊢ #p ] rs  % Variable
| [Γ ⊢ app M1 M2] ⇒
  let Arr ha f = main [Γ] [Γ ⊢ M1] rs in
  f [ ⊢ _ ] (main [Γ] [Γ ⊢ M2] rs)  % Application
| [Γ ⊢ lam λx. M1] ⇒
  Arr [ ⊢ h/value s/refl v/lam]
  (mlam N ⇒ fn rN ⇒ closed [ ⊢ s/beta]
   (main [Γ,x:tm _] [Γ,x ⊢ M1] (Cons rs rN)))  % Abstraction
| [Γ ⊢ c] ⇒ I [ ⊢ h/value s/refl v/c];  % Constant
```
Fundamental Lemma

```
rec closed : [ ⊢ mstep M M’] → Reduce [ ⊢ A] [ ⊢ M’] → Reduce [ ⊢ A] [ ⊢ M] = ? ;
rec main : {Γ:ctx}{M:[Γ ⊢ tm A]} RedSub [ ⊢ σ] → Reduce [ ⊢ A] [ ⊢ M σ] =
mlam Γ⇒mlam M ⇒ fn rs ⇒ case [Γ ⊢ M] of
| [Γ ⊢ #p ] ⇒ lookup [Γ] [Γ ⊢ #p ] rs % Variable
| [Γ ⊢ app M1 M2] ⇒
  let Arr ha f = main [Γ] [Γ ⊢ M1] rs in
  f [ ⊢ _ ] (main [Γ] [Γ ⊢ M2] rs)
| [Γ ⊢ lam λx. M1] ⇒
  Arr [ ⊢ h/value s/refl v/lam]
  (mlam N ⇒ fn rN ⇒ closed [ ⊢ s/beta]
   (main [Γ,x:tm _] [Γ,x ⊢ M1] (Cons rs rN)))
| [Γ ⊢ c] ⇒ I [ ⊢ h/value s/refl v/c]; % Constant
```

- Direct encoding of on-paper proof
- Equations about substitution properties automatically discharged
  (amounts to roughly a dozen lemmas about substitution and weakening)
- Total encoding about 75 lines of Beluga code
This Talk

Design and implementation of Beluga

- Introduction
- Example: Proof by logical relations
- Writing a proof in Beluga . . .
- Conclusion and current work
Revisiting the Design of Beluga

- **Top**: Functional programming with indexed types [POPL’08, POPL’12]
  - Case analysis
  - Inversion
  - Induction hypothesis
- **Bottom**: Contextual LF
  - On paper proof
  - Well-formed derivations
  - Renaming, Substitution
  - Well-scoped derivation
  - Context
  - Properties of contexts (weakening, uniqueness)
  - Substitutions (composition, identity)
  - In Beluga [IJCAR’10, CADE’15]
  - Dependent types
  - $\alpha$-renaming, $\beta$-reduction in LF
  - Contextual types and objects [TOCL’08]
  - Context schemas
  - Typing for schemas
  - Substitution type [LFMTP’13]
Introduction

Beluga: Design and implementation

Alternatives

General Theorem Proving Environments

- Calculus of Construction (Coq) / Martin Löf Type Theory (Agda)
  No special support for variables, assumptions, derivation trees, etc.
  About a dozen extra lemmas

- Isabelle / Nominal
  support for variable names, but not for assumptions, derivation trees, etc.
  based on nominal set theory; about a dozen extra lemmas
Alternatives

General Theorem Proving Environments

- **Calculus of Construction (Coq) / Martin Löf Type Theory (Agda)**
  No special support for variables, assumptions, derivation trees, etc.
  About a dozen extra lemmas

- **Isabelle / Nominal**
  Support for variable names, but not for assumptions, derivation trees, etc.
  Based on nominal set theory; about a dozen extra lemmas

Domain-specific Provers (Higher-Order Abstract Syntax (HOAS))

- **Abella**: encode second-order hereditary Harrop (HH) logic in $G$, an extension of first-order logic with a new quantifier $\nabla$, and develop inductive proofs in $G$ by reasoning about the size of HH derivations.
  Diverges a bit from on-paper proof; 4 additional lemmas

- **Twelf**: Too weak for directly encoding such proofs; implement auxiliary logic.
Current Work

- Prototype in OCaml (ongoing - last release March 2015) providing an interactive programming mode, totality checker [CADE’15]
  
  https://github.com/Beluga-lang/Beluga

- Mechanizing Types and Programming Languages - A companion:
  
  https://github.com/Beluga-lang/Meta

- Coinduction in Beluga (D. Thibodeau, A. Cave) Extending work on simply-typed copatterns [POPL’13] to Beluga Long term: reason about reactive systems [POPL’14]

- Case study: Certified compiler (O. Savary Belanger) [CPP’13]

- Extending Beluga to full dependent types (A. Cave)

- Type reconstruction (F. Ferreira [PPDP’14] and [JFP’13])

Thank you!

Download prototype and examples at

http://complogic.cs.mcgill.ca/beluga/

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“A language that doesn’t affect the way you think about programming, is not worth knowing.” - Alan Perlis