Preliminaries for the Completeness Proof:

Let Γ be a set. If P is a member of Γ we write:

$P \in \Gamma$.

If P is not a member of Γ we write:

$P \notin \Gamma$.

Example:

Let Γ be { A, B & Q, ~S }.

Then A $\in \Gamma$, B & Q $\in \Gamma$, ~S $\in \Gamma$, but B $\notin \Gamma$, Q $\notin \Gamma$, G $\notin \Gamma$.

Preliminaries for the Completeness Proof:

The Enumeration Lemma: (p. 238/258)

We can enumerate the set of *SL*-sentences. That is to say we can place the sentences in a one to one correspondence with the positive integers so we can talk about the first *SL*-sentence on our enumeration, and the $384{,}755{,}671{,}007{,}261{,}456{,}457{,}102{,}307{,}134^{th}$ *SL*-sentence on our enumeration, and so on.

See the text for the details (pp. 238-239/258-259).

Preliminaries for the Completeness Proof:

Maximal Consistency in *SD*: (p. 238/238)

A set Γ of *SL*-sentences is maximally consistent in *SD* if and only if Γ is consistent in *SD* and for every *SL*-sentence $P \notin \Gamma$, $\Gamma \cup \{ P \}$ is inconsistent in *SD*.

So, if a set of *SL*-sentences Γ is maximally consistent in *SD*, if we add to Γ any sentence which is not a member of it, the new augmented set will be inconsistent in *SD*.

If a set of *SL*-sentences Γ is maximally consistent in *SD*, we often say it is maximally consistent for short when the context makes it clear that it is maximal consistency with respect to *SD* that is in question.

Preliminaries for the Completeness Proof:

Lemma 6.4.1/6.4.2: (p. 237/257)

For any set of *SL*-sentences Γ and any sentence P, $\Gamma \models_{SL} P$ if and only if $\Gamma \cup \{ {\sim}P \}$ is truth-functionally inconsistent.

Proof:

See **L 6.3.4/6.3.5** (overhead 118, p. 233/249) and Exercise 3.6E.1c p. 100/113.

Lemma 6.4.2/6.4.4: (p. 237/257)

For any set of *SL*-sentences Γ and any sentence P, $\Gamma \vdash_{SD} P$ if and only if $\Gamma \cup \{ {\sim}P \}$ is inconsistent in *SD*.

Proof: See Exercise 6.4E.1

A “Picture” of the Logic of the Completeness Proof:

The Completeness Theorem is the single most important meta-result for Sentential Logic. Indeed, Gödel earned his PhD for proving the result in 1929. So. It is thus no surprise that the proof is the most complex one that we have seen so far.

We won't be following Gödel's method. Instead, our proof uses essentially the method that Leon Henkin employed in the 1950's. This has now become the standard method.

Henkin's proof is considerably easier to follow that many other presentations. That is is big advantage.

One disadvantage is that it is a non-constructive proof. What I mean by this is that it proves that if Γ truth-functionally entails P then there is a derivation of P from Γ. But it gives us no idea at all what that proof looks like. A constructive proof would give us a general method of constructing a derivation of P from Γ whenever Γ truth-functionally entails P. But the extra information a constructive proof provides comes at the cost of added complexity.

However, the Henkin-style proof is not without its difficulties. The next two slides give a graphical representation of the various steps.

L 6.2/6.4.5 Maximal Consistency Lemma:
If Γ is a set of *SL*-sentences that is consistent in *SD*, then Γ is a subset of a set of sentences that is maximally consistent in *SD*.

If Γ is a subset of Γ^* and Γ^* is truth-functionally consistent, then so is Γ.

L 6.3/6.4.8 The Consistency Lemma:
Every set of SL-sentences that is maximally consistent in *SD* is truth-functionally consistent

If Γ is consistent in *SD* then Γ is truth-functionally consistent.

By contraposition

L 6.1/6.4.3 The Inconsistency Lemma: If a set Γ of *SL*-sentences is truth-functionally inconsistent Γ is also inconsistent in *SD*.

By replacing "Γ" by "$\Gamma \cup \{ \sim P \}$"

If $\Gamma \cup \{ \sim P \}$ is truth-functionally inconsistent then $\Gamma \cup \{ \sim P \}$ is inconsistent in *SD*.

If $\Gamma \cup \{ \sim P \}$ is truth-functionally inconsistent then $\Gamma \cup \{ \sim P \}$ is inconsistent in *SD*.

L 6.4.2/6.4.5:
For any set of *SL*-sentences Γ and any sentence P, $\Gamma \vdash_{SD} P$ if and only if $\Gamma \cup \{ \sim P \}$ is inconsistent in *SD*.

If $\Gamma \cup \{\sim P \}$ is truth-functionally inconsistent then $\Gamma \vdash_{SD} P$.

L6.4.1:
For any set of *SL*-sentences Γ and any sentence P, $\Gamma \vDash_{SL} P$ if and only if $\Gamma \cup \{ \sim P \}$ is truth-functionally inconsistent.

$\Gamma \vDash_{SL} P \Rightarrow \Gamma \vdash_{SD} P$

As outlined, we prove Completeness by proving **L 6.1/6.4.3**.

We prove **L 6.1/6.4.3** by show how, for any **arbitrary** set of *SL*-sentences Γ that is consistent in *SD* how to construct a truth-value assignment on which all members of Γ are true. [This would show Γ to be truth-functionally consistent.]

We shall construct the truth-value assignment in two steps. If Γ is consistent in *SD*,

1) We form the set Γ^* which is a maximally consistent superset of Γ.
2) Having constructed Γ^* we show how to construct a truth-value assignment on which all members of Γ^* are true.

By constructing Γ^* from Γ we show **Lemma 6.2/6.4.5: The Maximal Consistency Lemma:**

If Γ is a set of *SL*-sentences that is consistent in *SD*, then Γ is a subset of at least one set of *SL*-sentences that is maximally consistent in *SD*.

Performing Step (2) will show **Lemma 6.3/6.4.8: The Consistency Lemma:**

Every set of *SL*-sentences that is maximally consistent in *SD* is truth-functionally consistent.

Sketch of a Proof of **Lemma 6.2/6.4.5: The Maximal Consistency Lemma:**

We start with a set of *SL*-sentences Γ that is consistent in *SD*.

We construct a superset of Γ, Γ^* by considering each sentence in our enumeration of the set of *SL*-sentences, adding it if and only if the resulting set is consistent in *SD*

1) Γ_1 is Γ.
2) If P_i is the *i*-th sentence in the enumeration then Γ_{i+1} is $\Gamma_i \cup \{P_i\}$ if $\Gamma_i \cup \{P_i\}$ is consistent in *SD*; otherwise Γ_{i+1} is Γ_i.

We then let Γ^* be the (infinite) union of all the Γ_i's. Γ^* is consistent in *SD.*

If it were not, then for some sentence P , both $\Gamma^* \vdash_{SD} P$ and $\Gamma^* \vdash_{SD} {\sim}P$.
But then by **L6.4.3/6.4.6** (p. 240/260)**:**

> If Γ is inconsistent in *SD*, then some finite subset of Γ is inconsistent in *SD*. (Proof is left as an exercise.)

one of the Γ_i's would be inconsistent in *SD*. But none of them are (by construction).

Similarly, Γ^* is maximal. We considered each *SL*-sentence in turn, and added it if and only if adding it would preserve consistency in *SD*. So no *SL*-sentence P such that $P \notin \Gamma^*$ can be added while preserving consistency in *SD*.

Completeness of *SD* Continued:

So we have Γ^* which is a maximally consistent extension of our original set Γ. (Recall that Γ was arbitrary.) If we can show how to construct a truth-value assignment on which all the members of Γ^* (and hence all those of Γ) are true, we will have show that a set that is consistent in *SD* is truth-functionally consistent. This will show **L6.1/6.4.3** and thus will show that *SD* is Complete.

Completeness of *SD* Continued:

We will need three more Lemmas:

Lemma 6.4.6/6.4.10 (p. 242/261)

If $\Gamma \cup \{P\}$ is inconsistent in *SD*, then $\Gamma \vdash_{SD} \sim P$.

Proof:

Left as an exercise (Ex 6.4.1)

Lemma 6.4.5/6.4.9 (p. 241/261)

If $\Gamma \vdash_{SD} P$ and Γ^* is a maximally consistent superset of Γ, then P is a member of Γ^*.

Proof:

Assume $\Gamma \vdash_{SD} P$ and Γ^* is a maximally consistent superset of Γ. Then $\Gamma^* \vdash_{SD} P$ too.

Now suppose, for a proof by contradiction, that P is not a member of Γ^*. Since Γ^* is maximal, it then follows from the definition of maximal consistency that $\Gamma^* \cup \{P\}$ is inconsistent in *SD*. By then, by **L 6.4.6/6.4.10**, it follows that $\Gamma^* \vdash_{SD} \sim P$. But then, $\Gamma^* \vdash_{SD} \sim P$ and $\Gamma \vdash_{SD} P$, but this is impossible, as Γ^* is maximally consistent in *SD* and hence consistent in *SD*. Thus, P is a member of Γ^* after all. ***QED***

Completeness of *SD* Continued:

L6.4.7/6.4.11 (p.242/262)

Let P and Q be *SL*-sentences. If Γ^* is maximally consistent then:

a) $P \in \Gamma^*$ if and only if $\sim P \notin \Gamma^*$.
b) $P \;\&\; Q \in \Gamma^*$ if and only if both $P \in \Gamma^*$ and $Q \in \Gamma^*$.
c) $P \vee Q \in \Gamma^*$ if and only if either $P \in \Gamma^*$ or $Q \in \Gamma^*$.
d) $P \supset Q \in \Gamma^*$ if and only if either $P \notin \Gamma^*$ or $Q \in \Gamma^*$.
e) $P \equiv Q \in \Gamma^*$ if and only if either $P \in \Gamma^*$ and $Q \in \Gamma^*$, or $P \notin \Gamma^*$ and $Q \notin \Gamma^*$.

(a)

$\Rightarrow$

Assume $P \in \Gamma^*$. Now assume for a proof by contradiction that $\sim P \in \Gamma^*$ as well. Then Γ^* has $\{ P, \sim P \}$ as a subset. So then Γ^* would be inconsistent in *SD*. But Γ^* is a maximal consistent set, so it is not inconsistent in *SD*—contradiction. Thus, $\sim P \notin \Gamma^*$.

$\Leftarrow$

Assume $\sim P \notin \Gamma^*$. Then, by the definition of maximal consistency in *SD*, $\Gamma^* \cup \{ \sim P \}$ is inconsistent in *SD*. So, Γ^* has a finite subset, call it Γ', such that $\Gamma' \cup \{ \sim P \}$ is inconsistent in *SD*. So, by **L 6.4.2/6.4.4** $\Gamma' \vdash_{SD} P$. Thus, by **L 6.4.5/6.4.9** $P \in \Gamma^*$.

L6.4.7/6.4.11 Continued:

(b) P & $Q \in \Gamma^*$ if and only if both $P \in \Gamma^*$ and $Q \in \Gamma^*$.

$\Rightarrow$

Assume P & $Q \in \Gamma^*$. Then { P & Q } is a subset of Γ^*. Then, since { P & Q } $\vdash_{SD} P$ and { P & Q } $\vdash_{SD} Q$, by **L 6.4.5/6.4.9**, $P \in \Gamma^*$ and $Q \in \Gamma^*$.

$\Leftarrow$

Assume $P \in \Gamma^*$ and $Q \in \Gamma^*$. Then, { P, Q } is a subset of Γ^*. Also, { P, Q } $\vdash_{SD} P$ & Q. Thus, by **L 6.4.5/6.4.9**, P & $Q \in \Gamma^*$.

(c) $P \vee Q \in \Gamma^*$ if and only if either $P \in \Gamma^*$ or $Q \in \Gamma^*$.
See Exercise 5.

(e) $P \equiv Q \in \Gamma^*$ if and only if either $P \in \Gamma^*$ and $Q \in \Gamma^*$, or $P \notin \Gamma^*$ and $Q \notin \Gamma^*$.
See Exercise 5.

L6.4.7/6.4.11 Continued:

(d) $P \supset Q \in \Gamma^*$ if and only if either $P \notin \Gamma^*$ or $Q \in \Gamma^*$.

$\Rightarrow$

Assume $P \supset Q \in \Gamma^*$. Either $P \notin \Gamma^*$ or $P \in \Gamma^*$. If $P \notin \Gamma^*$ then obviously $P \notin \Gamma^*$ or $Q \in \Gamma^*$. So assume $P \in \Gamma^*$. Then { P, $P \supset Q$ } is a subset of Γ^*. But { P, $P \supset Q$ } $\vdash_{SD} Q$ and thus by **L 6.4.5/6.4.9**, $Q \in \Gamma^*$. Thus, $P \notin \Gamma^*$ or $Q \in \Gamma^*$. Thus, whether $P \notin \Gamma^*$ or $P \in \Gamma^*$, $P \notin \Gamma^*$ or $Q \in \Gamma^*$.

$\Leftarrow$

Assume $P \notin \Gamma^*$ or $Q \in \Gamma^*$. If this is because $P \notin \Gamma^*$, then by (a), ~$P \in \Gamma^*$. So, either { ~P } or { Q } is a subset of Γ^*. But, either way, we can derive $P \supset Q$ from the relevant subset:

1	$\mid\ {\sim}P$	Ass
2	$\mid\ \mid\ P$	Ass
3	$\mid\ \mid\ \mid\ {\sim}Q$	Ass
4	$\mid\ \mid\ \mid\ P$	2R
5	$\mid\ \mid\ \mid\ {\sim}P$	1R
6	$\mid\ \mid\ Q$	3-5 ~E
7	$\mid\ P \supset Q$	2-6 $\supset$I

1	$\mid\ Q$	Ass
2	$\mid\ \mid\ P$	Ass
3	$\mid\ \mid\ Q$	1 R
4	$\mid\ P \supset Q$	2-3 $P \supset$E

Either way, $P \supset Q$ can be derived from a finite subset of Γ^*, so, by **L 6.4.5/6.4.9**, $P \supset Q \in \Gamma^*$.

This completes the proof of **L 6.4.7/6.4.11**. ***QED***

Completeness of *SD* Continued:

Proof of Lemma 6.3/6.4.8: The Consistency Lemma:

Let $\mathscr{A}^*$ be a truth-value assignment on which all of the atomic sentences that are members of Γ^* are assigned **T** and all other atomic sentences are assigned **F**.

We then show that for any *SL*-sentence P , $P \in \Gamma^*$ if and only if P is true on $\mathscr{A}^*$. We prove this by mathematical induction on the number of occurrences of sentential connectives in P. (This will show Γ^* to be truth-functionally consistent.)

<u>Basis Clause:</u>

Each atomic *SL*-sentence is a member of Γ^* if and only if it is true on $\mathscr{A}^*$.

Proof:

Obvious from definition of $\mathscr{A}^*$.

<u>Inductive Step:</u>

We show that

<u>If</u>

every *SL*-sentence with *k* or fewer occurrences of connectives is a member of Γ^* if and only if it is true on $\mathscr{A}^*$

<u>Then</u>

every *SL*-sentence with *k*+1 occurrences of connectives is a member of Γ^* if and only if it is true on $\mathscr{A}^*$.

Proof of Lemma 6.3/6.4.8 Continued:

Proof of the inductive step:

Assume the hypothesis of the induction (i.e. assume every *SL*-sentence with *k* or fewer occurrences of connectives is a member of Γ^* if and only if it is true on $\mathscr{A}^*$.)

We now show that for each possible sentence with *k*+1 occurrences of connectives is a member of Γ^* if and only if it is true on $\mathscr{A}^*$.

<u>Case 1:</u> P has the form ~Q .

If ~Q is true on $\mathscr{A}^*$, then Q is false on $\mathscr{A}^*$. Since Q contains *k* occurrences of connectives, the inductive hypothesis applies and $Q \notin \Gamma^*$. Then, by **L6.4.7(a)/6.4.11(a)**, ~$Q \in \Gamma^*$.

If ~Q is false on $\mathscr{A}^*$, then Q is true on $\mathscr{A}^*$. Since Q contains *k* occurrences of connectives, the inductive hypothesis applies and $Q \in \Gamma^*$. Then, by **L6.4.7(a)/6.4.11(a)**, ~$Q \notin \Gamma^*$.

<u>Case 2:</u> P has the form Q & R.

If Q & R is true on $\mathscr{A}^*$, then both Q and R are true on $\mathscr{A}^*$. Since they each contain at most *k* occurrences of connectives, the inductive hypothesis applies, and thus $Q \in \Gamma^*$ and $R \in \Gamma^*$. Then, by **L6.4.7(b)/6.4.11(b)**, Q & $R \in \Gamma^*$.

If Q & R is false on $\mathscr{A}^*$, then either Q or R is false on $\mathscr{A}^*$. Since they each contain at most *k* occurrences of connectives, the inductive hypothesis applies, and thus either $Q \notin \Gamma^*$ or $R \notin \Gamma^*$. Then, by **L6.4.7(b)/6.4.11(b)**, Q & $R \notin \Gamma^*$.

Proof of Lemma 6.3/6.4.8 Continued:

Cases 3 & 5:

3 and 5 are left as exercises, 4 is covered in the text, p. 244/262.

Case 4: P has the form $Q \supset R$.

If $Q \supset R$ is true on $\mathscr{A}^*$, then either Q is false on $\mathscr{A}^*$ or R is true on $\mathscr{A}^*$. Since they each contain at most *k* occurrences of connectives, the inductive hypothesis applies, and thus either $Q \notin \Gamma^*$ or $R \in \Gamma^*$. Then, by **L6.4.7(d)/6.4.11(d)**, $Q \supset R \in \Gamma^*$.

If $Q \supset R$ is true on $\mathscr{A}^*$, then Q is true on $\mathscr{A}^*$and R is false on $\mathscr{A}^*$. But then, by the inductive hypothesis (which applies since each of Q and R contain at most *k* occurrences of connectives), $Q \in \Gamma^*$ and $R \notin \Gamma^*$. Then, by **L6.4.7(d)/6.4.11(d)**, $Q \supset R \notin \Gamma^*$.

And this completes the proof of **L6.3/6.4.8**.

Completeness of *SD* Continued:

So, we have shown that a set that is maximally consistent in *SD* is truth-functionally consistent (we constructed a truth-value assignment upon which all of its members were true). But we have it that any arbitrary set Γ of *SL*-sentences that is consistent in *SD* can be extended to a maximally consistent set Γ^* (**L6.2/6.4.5**). Thus, any arbitrary set of *SL*-sentences Γ that is consistent in *SD* is truth-functionally consistent.

But this establishes **L6.1/6.4.3**. But we argued above that the Completeness of *SD* follows immediately from **L6.1/6.4.3**. Thus, allowing for the fact that we have merely sketched some fo the proofs, we have established **MT6.3/6.4.1** the Completeness of *SD*.

Q.E.D. **(Finalement!)**

Metatheorem 6.4/6.4.12: Compactness Theorem for *SL* (p. 245/265)

A set Γ of *SL*-sentences is truth-functionally consistent if and only if every finite subset of Γ is truth-functionally consistent.

Proof:

Exercise 9, p. 245/265.