Playing with big-Oh notation

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1 Big-Oh notation

Let \( f(n) \) and \( g(n) \) be two functions from the integers to the non-negative real numbers: \( f : \mathbb{N} \to \mathbb{R}^+ \), \( g : \mathbb{N} \to \mathbb{R}^+ \). We say that \( f(n) \) is \( O(g(n)) \) if and only if there exist constants \( c \in \mathbb{R} \) and \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( f(n) \leq c \cdot g(n) \). In more mathematical terms: \( f(n) \) is \( O(g(n)) \) iff: \( \exists c \in \mathbb{R}, N \in \mathbb{N} \) such that \( f(n) \leq c \cdot g(n) \) \( \forall n \geq N \).

To prove that a function \( f(n) \) is \( O(g(n)) \), one needs to find \( c \) and \( N \) such that \( f(n) \leq c \cdot g(n) \) \( \forall n \geq N \).

Example 1:
Let \( f(n) = 4n + 2 \) and \( g(n) = n \). Prove that \( f(n) \) is \( O(g(n)) \).
We want to find \( c \) and \( N \) such that \( 4n + 2 \leq c \cdot n \) \( \forall n \geq N \). Choosing \( c = 4 \) is not going to work. Let’s try \( c = 5 \). Then, we want to find \( N \) such that \( 4n + 2 \leq 5n \) \( \forall n \geq N \). Choosing \( N = 0 \) or \( N = 1 \) would not work. However, for \( N = 2 \), we have that if \( n \geq N \), then \( 4n + 2 \leq 4n + n = 5n = c \cdot n \). Thus, we have found constants \( c = 5 \) and \( N = 2 \) such that \( f(n) \leq c \cdot g(n) \) \( \forall n \geq N \). Consequently, \( f(n) \) is \( O(g(n)) \). Notice that we could have chosen \( c \) and \( N \) differently: \( c = 6, N = 1 \) also works. In fact, if \( f(n) \) is \( O(g(n)) \), there will be an infinite number of choices of \( c \) and \( N \) that will work.

Example 2:
With \( f(n) \) and \( g(n) \) defined as above, prove that \( g(n) \) is \( O(f(n)) \).
That’s easy: Pick \( c = 1, N = 1 \), then \( g(n) = n \leq c \cdot f(n) \) \( \forall n \geq N \).

Example 3:
Let \( f(n) = 3n + n \log_2(n) \) and \( g(n) = n \log_2(n) \). Prove that \( f(n) \) is \( O(g(n)) \).
Note: From here on, we will assume that \( \log(n) \) means \( \log_2(n) \). We want to find \( c \) and \( N \) such that \( 3n + n \log(n) \leq c n \log(n) \). How can we find them? Let’s set \( N = 2 \) (below \( N = 2 \), \( \log(n) < 1 \) which would cause trouble) and try to find a \( c \) that works. We would like \( 3n + n \log(n) \leq c n \log(n) \) \( \forall n \geq 2 \). Choosing \( c = 4 \) will work nicely: if \( n \geq 2 \), \( 3n + n \log(n) \leq 3n \log(n) + n \log(n) = 4n \log(n) = c g(n) \). Thus \( f(n) \) is \( O(g(n)) \). Notice that \( g(n) \) is also \( O(f(n)) \).
Example 4:
Let $f(n) = 2^{100}$ and $g(n) = 1$. Prove that $f(n)$ is $O(g(n))$.
We need to find $c$ and $N$ such that $2^{100} \leq c \cdot 1 \forall n \geq N$. That’s easy: pick $c = 2^{100}$ and $N = \text{anything}$.

Example 5:
Let $f(n) = 2^n + 8$ and $g(n) = 5^n + 2$. Prove that $f(n)$ is $O(g(n))$.
We need to find $c$ and $N$ such that $f(n) \leq c \cdot g(n) \forall n \geq N$. If we pick $c = 1$ and $N = 2$, then $n \geq N$ implies that $2^n + 8 = 2n + 6 + 2 \leq 2n + 3n + 2 = 5n + 2 = 1 \cdot g(n) \forall n \geq N$.
Now prove that $g(n)$ is $O(f(n))$.
Pick $c = 3$, $N = 1$. Then $g(n) = 5n + 2 \leq 3(2n + 8) = c \cdot f(n) \forall n \geq N$.

Notice: We have shown that $2^n + 8$ is $O(5^n + 2)$. In general, we will try to keep the function inside $O()$ the simplest as possible. Since being $O(5^n + 2)$ is equivalent to being $O(n)$, we will usually simply write that $2^n + 8$ is $O(n)$.

Proving that $f(n)$ is not $O(g(n))$
To prove that $f(n)$ is not $O(g(n))$, we must show that for any choice of $c$ and $N$, there exists an $n \geq N$ such that $f(n) > c \cdot g(n)$. Notice that the value of $n$ chosen will usually depend on $c$ and $N$.

Example 6
Prove that $n^2$ is not $O(n)$. Assume someone gives us a choice of $c$ and $N$. We must show that no matter what $c$ and $N$ are, we can find $n \geq N$ such that $n^2 > c \cdot n$. If we take $n = c + 1$, then $n^2 = (c + 1)^2 > c(c + 1) = cn$. However, we must ensure that $n$ is at least $N$, so let’s instead pick $n = \max(c + 1, N)$.

Example 7
Prove that $n^2$ is not $O(n \log(n))$. Given a choice of $c$ and $N$, we must exhibit an $n \geq N$ such that $n^2 > cn \log(n)$, or equivalently $n > c \log(n)$. Let’s try $n = c^c$. Then $c \log(n) = c \cdot c \log(c) = c^2 \log(c) < c^2 c = c^3 < c^c = n$, where the last inequality is true only if $c > 3$. But our proof has to work for any value of $c$. What if $c \leq 3$? In that case, simply pick $n = 16$, so that $n = 16 > 3 \log(16) = 12$. Now remember that $n$ chosen has to be at least $N$, so in conclusion, the choice of $n$ should be $\max(N, c^c)$ if $c > 3$ and $\max(N, 16)$ if $c \leq 3$. 

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