Playing with big-Oh notation

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1 Big-Oh notation

Let $f(n)$ and $g(n)$ be two functions from the integers to the non-negative real numbers: $f : \mathbf{N} \to \mathbf{R}^+$, $g : \mathbf{N} \to \mathbf{R}^+$. We say that $f(n)$ is $O(g(n))$ if and only if there exist constants $c \in \mathbf{R}$ and $N \in \mathbf{N}$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$. In more mathematical terms: $f(n)$ is $O(g(n))$ iff: $\exists c \in \mathbf{R}, N \in \mathbf{N}$ such that $f(n) \leq c \cdot g(n)) \ \forall n \geq N,$.

To prove that that a function $f(n)$ is $O(g(n))$, one needs to find c and N such that
$f(n) \leq c \cdot g(n) \ \forall n \geq N$.

Example 1:
Let $f(n) = 4n + 2$ and $g(n) = n$. Prove that $f(n)$ is $O(g(n))$.
We want to find c and N such that $4n + 2 \leq c \cdot n \ \forall n \geq N$. Choosing $c \leq 4$ is not going to work. Let's try $c = 5$. Then, we want to find N such that $4n+2 \leq 5n \ \forall n \geq N$. Choosing $N = 0$ or $N = 1$ would not work. However, for $N = 2$, we have that if $n \geq N$, then $4n + 2 \leq 4n + n = 5n = c \cdot n$. Thus, we have found constants $c = 5$ and $N = 2$ such that $f(n) \leq c \cdot g(n) \ \forall n \geq N$. Consequently, $f(n)$ is $O(g(n))$. Notice that we could have chosen c and N differently: $c = 6$, $N = 1$ also works. In fact, if $f(n)$ is $O(g(n))$, there will be an infinite number of choices of c and N that will work.

Example 2:
With $f(n)$ and $g(n)$ defined as above, prove that $g(n)$ is $O(f(n))$.
That's easy: Pick $c = 1$, $N = 1$, then $g(n) = n \leq c \cdot f(n) \ \forall n \geq N$.

Example 3:
Let $f(n) = 3n + n\log_2(n)$ and $g(n) = n\log_2(n)$. Prove that $f(n)$ is $O(g(n))$.
Note: From here on, we will assume that $\log(n)$ means $\log_2(n)$. We want to find c and N such that $3n + n\log(n) \leq cn\log(n)$. How can we find them? Let's set $N = 2$ (below $N = 2, \log(n) < 1$ which would cause trouble) and try to find a c that works. We would like $3n+n\log(n) \leq cn\log(n) \ \forall n \geq 2$. Choosing $c = 4$ will work nicely: if $n \geq 2$, $3n + n\log(n) \leq 3n\log(n) + n\log(n) = 4n\log(n) = cg(n)$. Thus $f(n)$ is $O(g(n))$. Notice that $g(n)$ is also $O(f(n))$.

Example 4:
Let $f(n) = 2^{100}$ and $g(n) = 1$. Prove that $f(n)$ is $O(g(n))$.
We need to find c and N such that $2^{100} \leq c \cdot 1 \ \forall n \geq N$. That's easy: pick $c = 2^{100}$ and $N = anything$.

Example 5;
Let $f(n) = 2n + 8$ and $g(n) = 5n + 2$. Prove that $f(n)$ is $O(g(n))$.
We need to find c and N such that $f(n) \leq cg(n) \ \forall n \geq N$. If we pick $c = 1$ and $N = 2$, then $n \geq N$ implies that $2n + 8 = 2n + 6 + 2 \leq 2n + 3n + 2 = 5n + 2 = 1 \cdot g(n) \ \forall n \geq N$.
Now prove that $g(n)$ is $O(f(n))$.
Pick $c = 3$, $N = 1$. Then $g(n) = 5n + 2 \leq 3(2n + 8) = c \cdot f(n) \ \forall n \geq N$.

Notice: We have shown that $2n+8$ is $O(5n+2)$. In general, we will try to keep the function inside $O()$ the simplest as possible. Since being $O(5n + 2)$ is equivalent to being $O(n)$, we will usually simpy write that $2n + 8$ is $O(n)$.

Proving that $f(n)$ is not $O(g(n))$
To prove that $f(n)$ is not $O(g(n))$, we must show that for any choice of c and N, there exists an $n \geq N$ such that $f(n) > c \cdot g(n)$. Notice that the value of n chosen will usually depend on c and N.

Example 6
Prove that n^2 is not $O(n)$. Assume someone gives us a choice of c and N. We must show that no matter what c and N are, we can find $n \geq N$ such that $n^2 > c \cdot n$. If we take $n = c + 1$, then $n^2 = (c + 1)^2 > c(c + 1) = cn$. However, we must ensure that n is at least N, so let's instead pick $n = \max(c + 1, N)$.

Example 7
Prove that n^2 is not $O(n \log(n))$. Given a choice of c and N, we must exhibit an $n \geq N$ such that $n^2 > cn \log(n)$, or equivalently $n > c \log(n)$. Let's try $n = c^c$. Then $c \log(n) = c \cdot c \log(c) = c^2 \log(c) < c^2 c = c^3 < c^c = n$, where the last inequality is true only if $c > 3$. But our proof has to work for any value of c. What if $c \leq 3$? In that case, simply pick $n = 16$, so that $n = 16 > 3 \log(16) = 12$. Now remember that n chosen has to be at least N, so in conclusion, the choice of n should be $\max(N, c^c)$ if $c > 3$ and $\max(N, 16)$ if $c \leq 3$.