Dynamic Programming Algorithms
Greedy Algorithms

Lecture 27

Return to Recursive algorithms: Divide-and-Conquer

- Divide-and-Conquer
  - Divide big problem into smaller subproblems
  - Conquer each subproblem separately
  - Merge the solutions of the subproblems into the solution of the big problem

Top-down approach

- Example:

Fibonnaci(n)

if (n ≤ 1) **then return** n

else return Fibonnaci(n-1) + Fibonnaci(n-2)

Very slow algorithm because we recompute Fibonnaci(i) many many times...

Dynamic programming

- Solve each small problem once, saving their solution
- Use the solutions of small problems to obtain solutions to larger problems

Bottom-up approach

```
FibonnaciDynProg(n)
int F[0...n];
F[0] = 0 ;
F[1] = 1;
for i = 2 to n do
   F[i] = F[i-2] + F[i-1]
return F[n]
```

The change making problem

- A country has coins worth 1, 3, 5, and 8 cents
- What is the smallest number of contains needed to make
  - 25 cents?
  - 15 cents?
- In general, with coins denominations C_1, C_2, ..., C_k, how to find the smallest number of coins needed to make a total of n cents?

Recursive algo. for making change

- Define Opt(n) as the optimal number of coins needed to make n cents
- We first write a recursive formula for Opt(n):

$Opt(0) = 0$

$Opt(n) = 1 + \min\{ Opt(n - C_1), Opt(n - C_2), \ldots Opt(n - C_k) \}$

(excluding cases where $C_i > n$)

Example: with coins 1, 3, 5, 8

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Opt(n)	1	2	1	2	1	2	3	1	2						

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Example: with coins 1, 3, 5, 8

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Opt(n)	1	2	1	2	1	2	3	1	2	2	2	3	2	3	

$Opt(15) = 1 + \min\{ Opt(15 - 1), Opt(15 - 3), Opt(15 - 5), Opt(15 - 8)\}$

$= 1 + \min\{ 3, 3, 2, 3 \}$

$= 1 + 2 = 3$

Recursive algo for making change

$\mathrm{Opt}(0) = 0$

$\mathrm{Opt}(n) = 1 + \min\{\ \mathrm{Opt}(\ n - C_1\ ), \mathrm{Opt}(\ n - C_2\ )\ , \ldots\ \mathrm{Opt}(\ n - C_k\ )\ \}$

(excluding cases where $C_i > n$)

Problem: Recursive algorithm is very slow, because it keeps recomputing the same Opt values over and over again

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Opt(n)	1	2	1	2	1	2	3	1	2	2	2	3	2	3	

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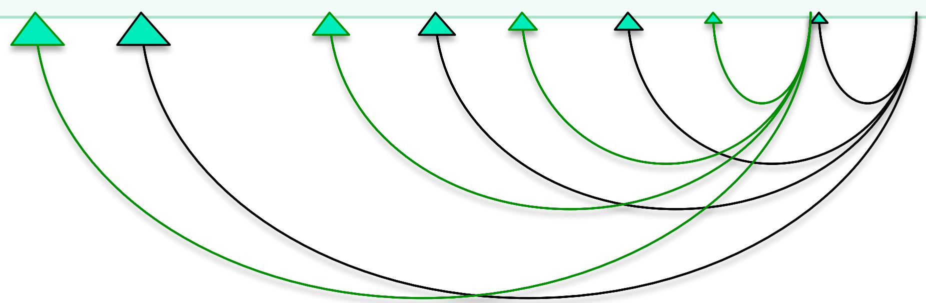
Opt(n)	1	2	1	2	1	2	3	1	2	2	2	3	2	3	



Recursive algo for making change

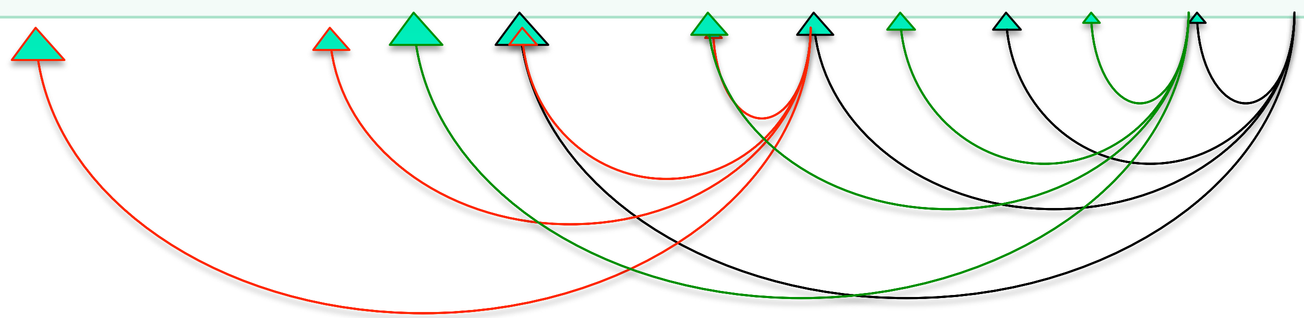
$$Opt(0) = 0$$

$$Opt(n) = 1 + \min\{\, Opt(\, n - C_1\,),\ Opt(\, n - C_2\,)\ ,\ \ldots\ Opt(\, n - C_k\,)\ \}$$

(excluding cases where $C_i > n$)

Problem: Recursive algorithm is very slow, because it keeps recomputing the same Opt values over and over again

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Opt(n)	1	2	1	2	1	2	3	1	2	2	2	3	2	3	



Recursive algo for making change

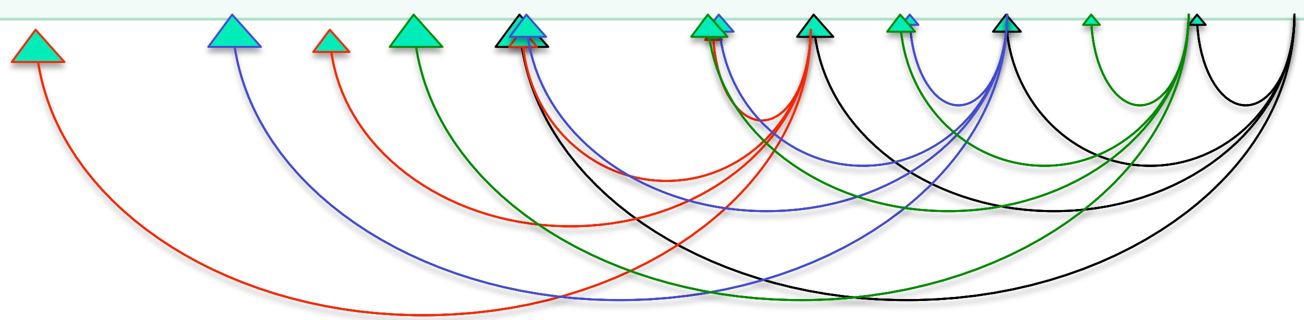
Opt(0) = 0

Opt(n) = 1 + min{ Opt(n – C_1), Opt(n – C_2) , … Opt(n – C_k) }

(excluding cases where C_i > n)

Problem: Recursive algorithm is very slow, because it keeps recomputing the same Opt values over and over again

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Opt(n)	1	2	1	2	1	2	3	1	2	2	2	3	2	3	



Dyn. Prog. Algo. for making change

- Use the same formula…

  $\mathrm{Opt}(0) = 0$

  $\mathrm{Opt}(n) = 1 + \min\{\ \mathrm{Opt}(\ n - C_1\ ), \mathrm{Opt}(\ n - C_2\ )\ , \ldots\ \mathrm{Opt}(\ n - C_k\ )\ \}$

  (excluding cases where $C_i > n$)

- But compute the values of Opt(i), starting with i=0, then i=1, … up to i=n. Save them in an array X

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X[n]	1	2	1	2	1	2	3	1	2	2	2	3	2	3	3

$$X[15] = 1 + \min\{\ X[15-1], X[15-3], X[15-5], X[15-8]\}$$
$$= 1 + \min\{\ 3, 3, 2, 3\ \}$$
$$= 1 + 2 = 3$$

Important: This is not a recursive algorithm!
Each entry in the array is computed *once*.

Algorithm makeChange(C[0..k-1], n)

Input: an array C containing the values of the coins

an integer n

Output: The minimal number of coins needed to make a total of n

```
int X[] = new int[n+1];       // X[0...n]
X[0] = 0
for i =1 to n do  // compute min{ Opt( i - C_j )}
    smallest = +∞
    for j = 0 to k-1 do
        if ( C[j] ≤ i ) then smallest=min(smallest, X[ i-C[j] ] )
    X[i] = 1 + smallest
Return X[n]
```

Making change - Greedy algorithm

- You need to give x ¢ in change, using coins of 1, 5, 10, and 25 cents. What is the smallest number of coins needed?
- Greedy approach:
  - Take as many 25 ¢ as possible, then
  - take as many 10 ¢ as possible, then
  - take as many 5 ¢ as possible, then
  - take as many 1 ¢ as needed to complete
- Example: 99 ¢ = 3* 25 ¢ + 2*10 ¢ + 1*5 ¢ + 4*1 ¢
- Is this always optimal?

Greedy-choice property

- A problem has the greedy choice property if:
  - An optimal solution can be reached by a series of locally optimal choices
- Change making: 1, 5, 10, 25 ¢: greedy is optimal
  1, 6, 10 ¢: greedy is not optimal

- For most optimization problems, greedy algorithms are not optimal. However, when they are, they are usually the fastest available.

Longest Increasing Subsequence

Problem: Given an array A[0..n-1] of integers, find the longest increasing subsequence in A.

Example: A = 5 1 4 2 8 4 9 1 8 9 2

Solution:

Slow algorithm: Try all possible subsequences…

for each possible subsequences s of A **do**
    if (s is in increasing order) **then**
        if (s is best seen so far) **then** save s
return best seen so far

Dynamic Programming Solution

Let LIS[i] = length of the longest increasing subsequence ending at position i and containing A[i].

A = 5 1 4 2 8 4 9 1 8 9 2

LIS =

LIS[0] = 1

$$LIS[i] = 1 + \max\{\ LIS[j] : j < i \ \text{ and } \ A[j] < A[i]\ \}$$

Dynamic Programming Solution

```
Algorithm LongestIncreasingSubsequence(A, n)
Input: an array A[0...n-1] of numbers
Output: the length of the longest increasing subsequence of A
LIS[0] = 1
for i = 1 to n-1 do
    LIS[i] = -1    // dummy initialization
    for j = 0 to i-1 do
        if ( A[j] < A[i] and LIS[i] < LIS[j]+1) then LIS[i] = LIS[j] + 1
return max(LIS)
```

Dynamic Programming Framework

- Dynamic Programming Algorithms are mostly used for optimization problems
- To be able to use Dyn. Prog. Algo., the problem must have certain properties:
  - **Simple subproblems**: There must be a way to break the big problem into smaller subproblems. Subproblems must be identified with just a few indices.
  - **Subproblem optimization**: An optimal solution to the big problem must always be a combination of optimal solutions to the subproblems.
  - **Subproblem overlap**: Optimal solutions to unrelated problems can contain subproblems in common.