Notes on Homology Theory

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We provide a short introduction to the various concepts of homology theory in algebraic topology. We closely follow the presentation in [3]. Interested readers are referred to this excellent text for a comprehensive introduction. We start with a quick review of some frequently used concepts of elementary group theory.

1 Free Abelian Groups

Let $(G_1, +)$ and $(G_2, *)$ be two Abelian groups. A map $f : G_1 \to G_2$ is said to be a *homomorphism* if

$$f(x + y) = f(x) * f(y),$$

for any $x, y \in G_1$. A bijective homomorphism is called an *isomorphism*. We write this as $G_1 \simeq G_2$. The *fundamental theorem of homomorphism* is stated as follows.

Theorem 1.1 *Let $f : G_1 \to G_2$ be a homomorphism. Then*

$$G_1/ker \simeq imf.$$

Take r elements $g_1, \cdots, g_r$ of a group G. The elements of G of the form

$$n_1 g_1 + \cdots + n_r g_r,\ n_i \in \mathbb{Z},\ 1 \le i \le r,$$

make a subgroup H inside G. H is said to be a subgroup generated $g_1, \cdots, g_r$. If G itself is generated by a finite number of elements, then G is said to be finitely generated. The elements $g_1, \cdots, g_r$ are said to be linearly independent if $n_1 g_1 + \cdots + n_r g_r = 0$ only when $n_1 = \cdots = n_r = 0$. If G is finitely generated by r linearly independent elements, G is called a *free Abelian group* of rank r.

If G is generated by one element g, $G = \{0, g, 2g, \cdots\}$ is called a *cyclic group*. If $ng = 0$ for some $n \in \mathbb{Z} - \{0\}$, then G is a *finite cyclic group*. Otherwise, it is an *infinite cyclic group*. Any infinite cyclic group is isomorphic to $\mathbb{Z}$, while a finite cyclic group is isomorphic to some $\mathbb{Z}_n$. If

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G is free Abelian group of rank r and H is subgroup of G. We may choose p generators $g_1, \cdot, g_p$ out of r generators of G so that $k_1 g_1, \cdots, k_p g_p$ generate H of rank p. In other words

$$H \simeq k_1\mathbb{Z} \oplus k_2\mathbb{Z} \oplus \cdots \oplus k_p\mathbb{Z}.$$

We now give the *fundamental theorem of finitely generated Abelian groups*.

Theorem 1.2 *Let G be a finitely generated Abelian group (not necessarily free) with m generators. Then G is isomorphic to the direct sum of cyclic groups,*

$$G \simeq \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p},$$

where $m = r + p$. r is called the rank of G.

Finally an *exact sequence* is defined as a sequence of Abelian groups and homomorphisms between them,

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

such that $\ker \alpha_n = \mathrm{im}\ \alpha_{n+1}$ for each n.

2 Topological spaces and Homotopy

Let X be any set and J be an index set. Let $\mathcal{U} = \{U_i \mid i \in J\}$ denote a certain collection of open subsets of X. The pair $(X, \mathcal{U})$ is a *topological space* of $\mathcal{U}$ satisfies the following:

1. $\emptyset, X \in \mathcal{U}$.
2. If I is a (possibly infinite) sub-collection of J, then $\cup_{i \in I} U_j \in \mathcal{U}$.
3. If K is any finite sub-collection of J, then $\cap_{k \in K} U_k \in \mathcal{U}$.

A *deformation retract* of a topological space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in [0,1]$, such that $f_0 = \mathrm{id}$ (the identity map), $f_1(X) = A$ and $f_t|_A = \mathrm{id}$ for all t. The family f_t, should be continuous in the sense that the associated map $X \times [0,1] \to X$, $(x,t) \mapsto f_t(x)$ is continuous.

A deformation retract is a special case of the general notion of homotopy. A *homotopy* is simply any family of maps $f_t : X \to Y$, $t \in [0,1]$, such that the associated map $F : X \times [0,1] \to Y$ given by $F(x,t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \to Y$ are said to be *homotopic maps*, if there exists a homotopy f_t connecting them and once writes $f_0 \simeq f_1$. In these terms a deformation retract of X onto a subspace A is a homotopy from the identity map of X onto A, a map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathrm{id}$.

In this work, we are mainly concerned with a special type of topological spaces, known as simplicial complexes. For an introduction to simplicial complexes, see Chapter **??**. Here, we introduce some broader classes of topological spaces, namely the CW complexes and Δ-complexes. Simplicial complexes are special types of these spaces. We therefore present the general theory for a more comprehensive introduction.

A *cell complex* is a topological space constructed by the following procedure:

1. Start with a discrete set X^0, whose points are regarded as 0-cells.
2. Inductively, from the *n-skeleton* X^n, construct X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. Once can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is give the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

Cell complexes are also called as CW complexes. An example of a cell complex is drawn in Figure 1. This cell complex has one 0-cell, two 1-cells and one 2-cell. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$.

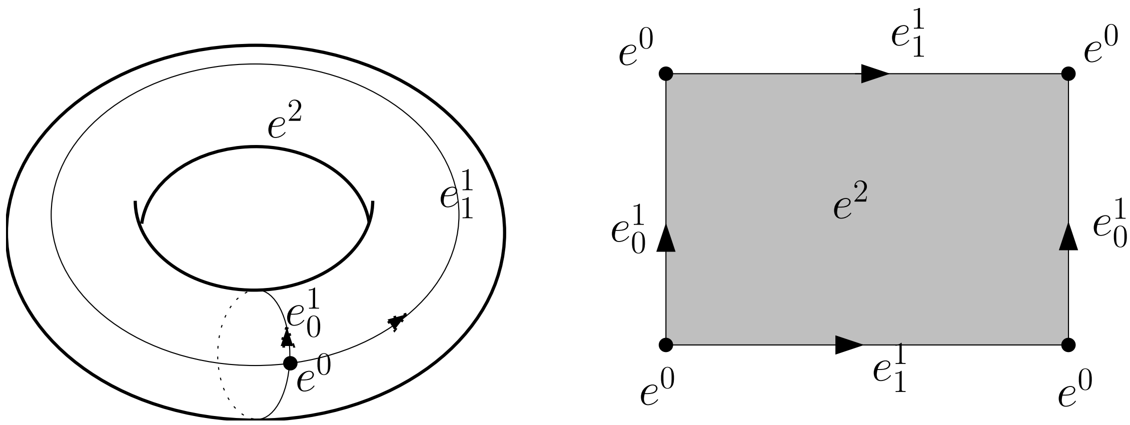

Figure 1: A cell complex representation of a torus $S^1 \times S^1$.

Each n-cell e^n_α in a cell complex has a *characteristic map* $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map ϕ_α and is a homomorphism from the interior of D^n_α onto e^n_α. Therefore, Φ_α can be thought of as the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \xrightarrow{\pi} X^n \hookrightarrow X.$$

where π is the quotient map defining X^n.

A *sub-complex* of a cell complex is a closed subspace $A \subset X$ that is a union of cells of X. A is a cell complex in its own right. A pair (X, A) consisting of a cell complex X and a sub-complex A is called a *CW pair*.

A graph is a 1-dimensional cell complex. It contains vertices (0-cells) and edges (1-cells). Similarly, simplicial complexes can also be thought of as cell complexes. It is however, more instructive to start with a more primitive form of complexes, known as Δ-complexes.

Δ-complexes are built out of simplices. An *n-simplex* is defined as the smallest convex set in $\mathbb{R}^d$ containing $n+1$ points $v_0, \cdots, v_n$, that do not lie in a hyperplane of dimension less than n. The points v_i are called the *vertices* of the simplex, and the simplex itself is denoted by $[v_0, v_1, \cdots, v_n]$. The standard n-simplex is given by

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, \text{ and } t_i \geq 0 \text{ for all } i\}.$$

A *face* of a simplex $[v_0, \cdots, v_n]$ is the sub-simplex with vertices any nonempty subset of the v_i's. By convention, a face is ordered according to their order in the larger simplex. A Δ*-complex* X, is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices. Hence, the identifications never result in two distinct points in the interior of a face, being identified in X. Therefore, X is the disjoint union of a collection of open simplices (simplices with their proper faces deleted).

Each such open simplex e^n_α of dimension n comes equipped with a canonical map (called the characteristic map) $\sigma_\alpha : \Delta^n \to X$ restricting to a homeomorphism from the interior of Δ^n onto e^n_α. A key property of the characteristic map is that its restrictions to $(n-1)$-dimensional faces of Δ^n are characteristic maps σ_β for open simplices e^{n-1}_β of X. This property can be used to define a Δ-complex as a CW complex X in which each n-cell e^n_α has a distinguished characteristic map $\sigma_\alpha : \Delta^n \to X$ such that the restriction of σ_α to each $(n-1)$-face of Δ^n is the distinguished characteristic map of an $(n-1)$-cell of X.

3 Simplicial Homology

We first define the simplicial homology of Δ-complexes. For a more gentle introduction, see Chapter **??**. Let $\Delta_n(X)$ ne the free Abelian group with basis the open simplices e^n_α of the Δ-complex X. The elements of $\Delta_n(X)$ are called as n-chains. These elements can be written as finite sums $\sum_\alpha n_\alpha e^n_\alpha$ with coefficients $n_\alpha \in \mathbb{Z}$. One can also consider them as $\sum_\alpha n_\alpha \sigma_\alpha$.

The *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ can be defined by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}.$$

Lemma 3.1 *The composition* $\Delta_n(X) \xrightarrow{\partial_n} \Delta_n(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ *is zero. In other notation* $\partial_n \circ \partial_{n-1} = 0$.

Proof: This can be checked by a simple calculation.

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\left(\sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_n]}\right) \\
&= \sum_{j<i} (-1)^i(-1)^j \sigma|_{[v_0,\cdots,\hat{v_j},\cdots,\hat{v_i},\cdots,v_n]} + \sum_{j>i} (-1)^i(-1)^{j-1} \sigma|_{[v_0,\cdots,\hat{v_i},\cdots,\hat{v_j},\cdots,v_n]} \\
&= 0.
\end{aligned}$$

The chain groups $\Delta_n(X)$ are generally denoted by C_n. Note that each of the chain groups C_n is an Abelian group. We therefore get a sequence of homomorphisms of Abelian groups

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_k \partial_{k+1} = 0$ for each k. Such a sequence is called a *chain complex*. From $\partial_k \partial_{k+1} = 0$ it follows that $\mathrm{im}\partial_{n+1} \subset \ker\partial_n$. We define the *simplicial homology groups* by the quotient groups

$$H_n^{\Delta}(X) = \frac{\ker\partial_n}{\mathrm{im}\partial_{n+1}}.$$

The elements of $H_n^{\Delta}(X)$ are the cosets of $\mathrm{im}\partial_{n+1}$, and are referred to as *homology classes*. Elements of $\ker\partial_n$ are called as *cycles* and those of $\mathrm{im}\partial_{n+1}$ are called as *boundaries*. Two cycles representing the same homology class are said to be *homologous*.

4 Example Computations of Simplical Homology

The dimension of $H_0^{\Delta}(X)$, is equal to the number of path-connected components of X. The simplest basis for $H_0^{\Delta}(X)$ consists of a choice of vertices in X, one in each path-component of X. Likewise, the simplest basis for $H_1^{\Delta}(X)$ consists of loops in X, each of which surrounds a different 'hole' in X. For example, if X is a graph, then $H_1^{\Delta}(X)$ is a measure of the number and types of cycles in the graph. These concepts can be understood more clearly with the following example.

In Figure 2, a hollow doughnut-like two-dimensional surface, called a torus has been drawn. Imagine that we cut this torus at the edges a and b, as depicted in the Figure. We flatten the resulting surface on a plane, triangulate and label it as shown in the Figure. The resulting triangulation is a valid Δ-complex. It is made of one 0-simplex v, three 1-simplices a, b and c and two 2-simplices U and L. The arrows on the simplices indicate the orientations on the simplices. Finally, note that it is possible to assemble a torus from this simplicial complex by the identification of the multiple edges a and b, centered at v.

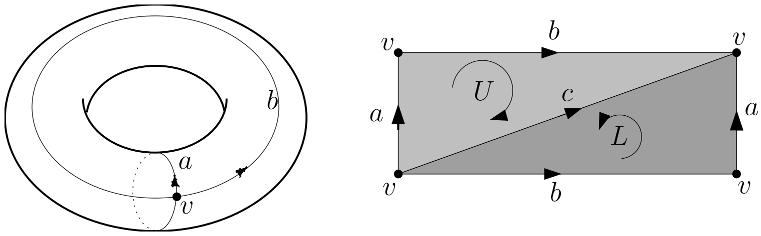

Figure 2: A torus [Left] and a Δ-complex [Right] corresponding to its triangulation T.

Let us start with the zero-th homology group. With only one vertex v, $C_0(T) \simeq \mathbb{Z}$. Similarly, $C_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, indicating the free group on the three edges a, b, c. Any $c \in C_1(T)$ can be expressed as $c = \alpha a + \beta b + \gamma c$, where $\alpha, \beta, \gamma \in \mathbb{Z}$. Clearly, the boundary map $\partial_1 : C_1(T) \to C_0(T)$ is zero. To see this, note that $\partial_1(c) = \alpha\partial_1(a) + \beta\partial_1(b) + \gamma\partial_1(c) = \alpha(v-v) + \beta(v-v) + \gamma(v-v) = 0$. Therefore,

$$H_0^\Delta(T) \simeq \ker\partial_0/\mathrm{im}\partial_1 \simeq C_0(T) \simeq \mathbb{Z}.$$

This is consistent with the observation that the space has only connected component.

Consider another basis for $C_1(T)$ as $\{a, b, a+b-c\}$. Since $\partial_2(U) = \partial_2(L) = a + b - c$, it follows that

$$H_1^\Delta(T) \simeq \ker\partial_1/\mathrm{im}\partial_2 \simeq \mathbb{Z} \oplus \mathbb{Z},$$

by modding out the component in the free group $C_1(T)$, corresponding to $a+b-c$. This leaves a and b as the representative cycles for the two non-trivial homology classes in the first homology group.

Since there are no simplices for dimension 3 or higher, $C_k(T) \simeq 0$ for $k > 2$. From this it follows that

$$H_2^\Delta(T) \simeq \ker\partial_2/\mathrm{im}\partial_3 \simeq \ker\partial_2 \simeq \mathbb{Z}.$$

This is the free group generated by $U - L$, since for any $\alpha U + \beta L \in C_2(T)$, $\partial_2(\alpha U + \beta L) = (\alpha + \beta)(a + b - c) = 0$ if and only if $\alpha = -\beta$. Finally, $H_k^\Delta(T) \simeq 0$ for $k > 2$. To summarize,

$$H_n^\Delta(T) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1; \\ \mathbb{Z}, & \text{for } n = 0.2; \\ 0, & \text{for } n \geq 0. \end{cases}$$

Note that each Δ-complex can be transformed into a simplicial complex (the likes of which we have encountered in this work). This can be done using a technique called as barycentric subdivision. It can be shown that the second barycentric subdivision of any Δ-complex produces a simplicial complex, which is homeomorphic to the original Δ-complex. Without going into details, it is enough to understand that the simplicial complexes are Δ-complexes whose simplices are uniquely determined by their vertices. In Δ-complexes, this restriction is not in force. This is the reason that the Δ-complex representation of the torus drawn in Figure 2 has only two 2-simplices. A simplicial complex representation of the torus, however, would require at least 14 triangles, 21 edges and 7 vertices. The Δ-complex representation, therefore makes the computations much easier in many cases.

5 Singular Homology and Homotopy Invariance

Simplicial homology is a very powerful theory. However, there is a more elegant homology theory, known as singular homology, which lets us study many questions in a more straightforward manner. Many of the results developed in singular homology also carry for Δ-complexes and in many cases for simplicial complexes as well. We introduce this theory below.

A *singular n-simplex* in a space X is a continuous map (instead of a set) given by $\sigma : \Delta^n \to X$. With the set of all such singular n-simplices as a basis, one can generate a free Abelian group $C_n(X)$, whose elements are called the *chains*. Each n-chain can be written as a finite formal sum $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. Similarly the boundary map between singular chains $\partial_n : C_n(X) \to C_n(X)$ is given by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_n]}.$$

A similar proof to the one presented for simplicial homology shows that $\partial_n\partial_{n+1} = 0$. Therefore, the singular homology groups are given by

$$H_n(X) = \text{ker } \partial_n/\text{im } \partial_{n+1}.$$

On the face of it, the singular homology theory looks very similar to simplicial homology. However, there are many subtle differences. We briefly summarize some results from singular homology theory which may have some analogs in simplicial homology, but are easier to derive in the singular theory.

If of a space X has path-wise connected components X_α, then

$$H_n(X) \simeq \oplus_\alpha H_n(X_\alpha).$$

If X is path-wise connected, then $H_0(X) \simeq \mathbb{Z}$. For a space with multiple components, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s for each component of X. If X is homotopic to a point, then $H_n(X) = 0$ for $n > 0$. For detailed proofs, please see [3].

We now present a result which is particularly important for this work: Spaces that are homotopy equivalent have isomorphic homology groups.

Corresponding to each map between spaces $f : X \to Y$, there is an induced homomorphism between their respective chain groups denoted by $f_\sharp : H_n(X) \to H_n(Y)$ for each n. This can be defined in the following way. Since each singular n-simplex is given by $\sigma : \Delta^n \to X$, we compose it with f to get

$$f_\sharp(\sigma) = f\sigma : \Delta^n \to Y.$$

This can extended linearly over any chain in $C_n(X)$ to get

$$f_\sharp(\sum_i n_i\sigma_i) = \sum_i n_i f_\sharp(\sigma_i) = \sum_i n_i f\sigma_i.$$

Let us now see how this maps behaves with the boundary operators.

$$\begin{aligned} f_\sharp\partial(\sigma) &= f_\sharp\left(\sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_n]}\right), \\ &= \sum_i (-1)^i f\sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_n]} \\ &= \partial f_\sharp(\sigma). \end{aligned}$$

This means that $f_\sharp\partial = \partial f_\sharp$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \to & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n+1}(X) & \to \cdots \\ & & \downarrow f_\sharp & & \downarrow f_\sharp & & \downarrow f_\sharp & \\ \cdots & \to & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n+1}(Y) & \to \cdots \end{array}$$

Consider a cycle α, i.e. $\partial\alpha = 0$. Then

$$\partial(f_\sharp\alpha) = f_\sharp(\partial\alpha) = 0.$$

In other words, $f_\sharp$ takes cycles in X to cycles in Y. Also, if $\partial\beta$ is a boundary in X,

$$f_\sharp(\partial\beta) = \partial(f_\sharp\beta),$$

which is boundary in Y. This proves that $f_\sharp$ is a *chain map*, i.e. it induces a homomorphism between the respective homology groups

$$f_* : H_n(X) \to H_n(Y),$$

which satisfies two elementary properties

1. The identity map id : $X \to X$ induces the identity map id_* on the homology groups.
2. The composition of two maps $X \xrightarrow{g} Y \xrightarrow{f} Z$ induces the composition of the induced homomorphisms: $(gf)_* = g_* f_*$.

Finally, we give the following result.

Theorem 5.1 *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

For a detailed proof of this theorem, we refer the reader to [3]. The main ingredient of the proof is a method of subdividing $\Delta^n \times [0,1]$ into $n+1$ simplices and the use of a certain prism operator P as a chain homotopy between $g_\sharp$ and $f_\sharp$. First, the following relation is derived.

$$\partial P = g_\sharp - f_\sharp - P\partial.$$

Then, consider a cycle $\alpha \in C_n(X)$. Since $\partial\alpha = 0$, we have

$$g_\sharp(\alpha) - f_\sharp(\alpha) = \partial P(\alpha) + P\partial(alpha) = \partial P(\alpha).$$

This means that $g_\sharp(\alpha) - f_\sharp(\alpha)$ is a boundary, which means that both $g_\sharp(\alpha)$ and $f_\sharp(\alpha)$ define the same homology class. Therefore $f_*(\alpha) = g_*(\alpha)$, proving the theorem.

From these properties of f_*, g_* we immediately get our main result.

Corollary 5.1 *The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all n.*

6 Relative Homology Groups and Exact Sequences

Relative homology groups are useful tools for studying quotient spaces. Let A be a subspace of a space X and denote by $C_n(X, A)$ the quotient chain group $C_n(X)/C_n(A)$. Therefore any chain inside A is considered to be trivial in $C_n(X, A)$. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ induces a quotient boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. Here too, $\partial_n\partial_{n+1} = 0$ holds. Therefore one can define the *relative homology groups*, $H_n(X, A) = \ker \partial_n / \mathrm{im}\ \partial_{n+1}$ using these boundary operators in exactly the same manner. It should be noted that

1. Elements of $H_n(X, A)$ are called *relative cycles*. They are n-chains $\xi \in C_n(X)$ such that $\partial\xi \in C_{n-1}(A)$.
2. A cycle α in called *trivial*, if it is a *relative boundary*. In other words, $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

It can be shown that these chain groups satisfy the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \to & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X, A) & \to 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
0 & \to & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{j} & C_{n-1}(X, A) & \to 0
\end{array}$$

where i is the inclusion map and j is a quotient map with respect to A. From this, it can be shown that relative homology groups $H_n(X, A)$ for any pair $(X, A \subset X)$ satisfy the *long exact sequence*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X, A) \to 0$$

(Recall the definition of exactness from the first section on Abelian groups). Notice, that this sequence is defined for *any* pair (X, A). One might wonder, as to why not define the homology groups for the quotient space X/A directly. It can be shown that we do have a long exact sequence,

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

However, the existence of such a sequence requires that (X, A) is a *good pair*, namely that A is a non-empty closed subspace that is a deformation retract of some neighborhood in X. The long exact sequence for the relative homology groups, however, holds for any pair, and is therefore preferred over the the exact sequence for the homology of the quotient space X/A.

Finally, it is appropriate to mention the equivalence of simplicial and singular homology for a Δ-complex X. One can define a homomorphism $\theta : \Delta_n(X) \to C_n(X)$ between the two chain groups by sending each n-simplex of X to its characteristic map $\sigma : \Delta^n \to X$. From this one can get a canonical homomorphism between the respective homology groups. One can prove the following general result.

Theorem 6.1 *The induced homomorphisms, $H_n^\Delta(X, A) \to H_n(X, A)$, are isomorphisms for all n and all Δ-complex pairs (X, A).*

For a detailed proof we refer the reader to [3].

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