Reflection Algebra in Two and Three Dimensions

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1. Fundamental reflections.

The two fundamental rotations, R_0 and R_{90}, can be abstracted as the numbers 1 and i, from which any rotation by angle c, s (with $c^2 + s^2 = 1$)

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = c\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + s\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \rightarrow c + is$$

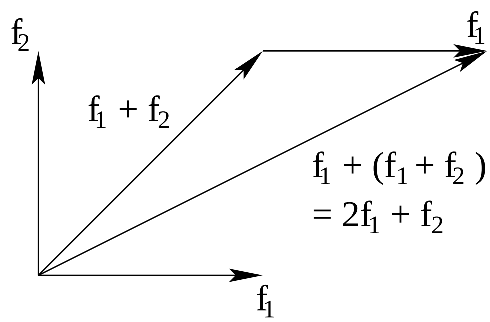
giving us, with the single additional rule that $i^2 = -1$, the "two-dimensional numbers".

Similarly the reflection in a line at *half* the angle, $\frac{1}{2}\angle(c, s)$, can be decomposed into two fundamental reflections

$$\begin{pmatrix} c & s \\ s & -c \end{pmatrix} = c\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + s\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \rightarrow cf_1 + sf_2$$

These are reflections in lines at 0 degrees and 45 degrees, respectively, yet f_1 and f_2 can be interpreted as axes at *right angles* to each other.

If we add $f_1 + f_2$, for instance, we get a 45-degree line. Adding f_1 to that has the effect of appending an f_1 to the end of $f_1 + f_2$ and is also just $2f_1 + f_2$. This is exactly like adding components in two dimensions.



Note that f_1 and f_2 in this interpretation are *intervals* in two-dimensional space. They do not have absolute positions but can be located anywhere in the space.

When we get to multiplication we have some new algebraic properties.

First, since any reflection repeated is just the identity, we have

$$f_1^2 = 1 \qquad \text{and} \qquad f_2^2 = 1$$

Second, since two reflections make a rotation (at twice the angle separating the mirrors), we have a new symbol

$$f_{12} = f_1 f_2 \leftarrow \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -R_{90}$$

Thus

$$f_{12}^2 = -1$$

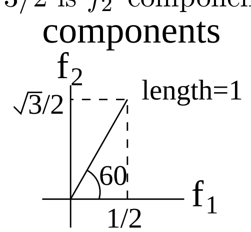
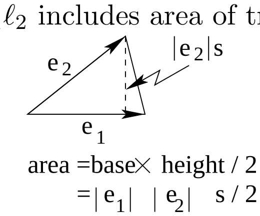
just as $i^2 = -1$ (since i is an alias for R_{90}). Furthermore

$$f_{21} = f_2 f_1 = R_{90} = -f_{12}$$

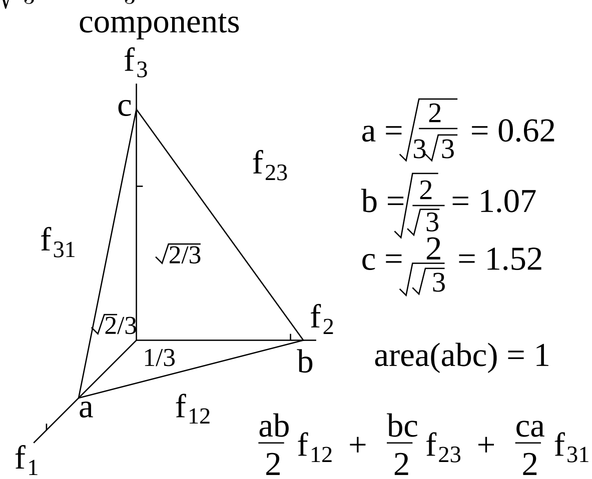
As with matrices, and since order now matters, we will have a convention of working from right to left.

Unlike 2-D numbers, this works in any number of dimensions. Reflections are more fundamental than rotations.

2. Reflection algebra.

1	$f_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $f_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	2-D basic reflections; $f_1^2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = f_2^2$, $f_1 f_2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = -\mathrm{rot}(90)$ $f_2 f_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \mathrm{rot}(90)$
2	$f_1, f_2, \cdots, f_d$	basis for d-dimensional space $f_j^2 = 1$, $f_{jk} = f_j f_k$, $f_{jk}^2 = -1$ for $j \neq k$
3	ℓ: $\ell_1, \ell_2, \cdots$	line, normalized, $\ell^2 = 1$; subscripted for different lines e.g., $\ell = \frac{1}{2} f_1 + \frac{\sqrt{3}}{2} f_2$: 1/2 is f_1-component, $\sqrt{3}/2$ is f_2–component 
4	e: $e_1, e_2, \cdots, e_d$	edge, e = length×corresponding ℓ, length$^2 = e^2$, $\mid e \mid$ = length of e e.g., $e = f_1 + \sqrt{3} f_2$, $e^2 = 1 + 3 = 4$, $\mid e \mid = \sqrt{4} = 2$
5	ℓe	maps $\ell \rightarrow e$, because $\ell(\ell e) = (\ell\ell)e = e$
6	$\ell_1 \ell_2 = c + sn$	rotates $\ell_1 \rightarrow \ell_2$: angle c, s in plane n (see 10) with $c^2 + s^2 = 1$ NB $e_1 e_2 = \mid e_1 \mid\mid e_2 \mid \ell_1 \ell_2$ includes area of triangle 
7	$\ell e \ell$	invert e in ℓ: $\parallel$ same sign, $\perp$ change sign e.g., $f_1(f_1 + f_2 + f_3)f_1 = f_1 - f_2 - f_3$
8	$\frac{1}{2}(e + \ell e \ell)$ $\frac{1}{2}(e - \ell e \ell)$	projection of e onto ℓ projection of $e \perp \ell$
9	$e f_{12}$ $e(c + s f_{12})$	in 2D, perpendicular to e, e.g., $(cf_1 + sf_2)f_{12} = -sf_1 + cf_2$ rotate e, e.g., $(c' f_1 + s' f_2)(c + s f_{12}) = (cc' + ss')f_1 + (c's + cs')f_2$, new angle $= \angle(c, s) + \angle(c', s')$

10	n: $n_1, n_2, \cdots$	plan*e*, normalized $n^2=-1$; subscripted for different planes e.g., $n = \frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}$



11	h: $h_1, h_2, \cdots$	face (hedron); h= area$\times$corresponding n, area^2 = h^2, $\mid h \mid$ = area of h e.g., $h = f_{12} + \sqrt{6}f_{23} + \sqrt{2}f_{31}$, $h^2 = 1+6+2 = 9$, $\mid h \mid = \sqrt{9} = 3$
12	$-nh$	maps $n \to h$ because $n(-nh) = (-nn)h = h$
13	$n_1n_2 = c + sn_3$	rotates $n_1 \to n_2$, "dihedral" angle c, s $(c^2+s^2=1)$ in plane n_3, $n_3 \perp n_1$, $n_3 \perp n_2$
14	ℓn $-n\ell$	maps $\ell \to n$ maps $n \to \ell$
15	$-nhn$	invert h in n, $\parallel$ same sign, $\perp$ change sign e.g., $-f_{12}(f_{12}+f_{23}+f_{31})f_{12} = f_{12} - f_{23} - f_{31}$
16	$\frac{1}{2}(h - nhn)$ $\frac{1}{2}(h + nhn)$	projection of h onto n projection of $h \perp n$
17	$\ell h \ell$	invert h in plane $\perp \ell$, e.g., $f_1(f_{12}+f_{23}+f_{31})f_1 = -f_{12}+f_{23}-f_{31}$
18	$\frac{1}{2}(h - \ell h \ell)$ $\frac{1}{2}(h + \ell h \ell)$	projection of h off $\perp \ell$ projection of $h \perp \ell$
19	nen	invert e in n, e.g., $f_{12}(f_1+f_2+f_3)f_{12} = f_1+f_2-f_3$
20	$\frac{1}{2}(e + nen)$ $\frac{1}{2}(e - nen)$	projection of e onto n projection of $e \perp n$
21	ef_{123}	in 3D, perpendicular to e, e.g., $(pf_1+qf_2+rf_3)f_{123} = rf_{12}+pf_{23}+qf_{31}$
22	$-hf_{123}$	in 3D, perpendicular to h, e.g., $-(pf_{12}+qf_{23}+rf_{31})f_{123} = qf_1+rf_2+pf_3$
23	$\ell_2\ell_1e\ell_1\ell_2$	rotates e in plane of ℓ_1, ℓ_2 by twice angle between ℓ_1, ℓ_2 e.g. (see 3), $(\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2)f_1(f_1+f_2+f_3)f_1(\frac{1}{2}f_1+\frac{\sqrt{3}}{2}f_2)$ $= (\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12})(f_1+f_2+f_3)(\frac{1}{2}+\frac{\sqrt{3}}{2}f_{12})$ $= (\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4})f_1 + (\frac{1}{4}+\frac{3}{4}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4})f_2 + (\frac{1}{4}+\frac{3}{4})f_3 + (-\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4})f_{123}$ $= (-\frac{1}{2} - \frac{\sqrt{3}}{2})f_1 + (-\frac{1}{2}+\frac{\sqrt{3}}{2})f_2 + f_3$ $= (-\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2) + (-\frac{1}{2}f_2 - \frac{\sqrt{3}}{2}f_1) + f_3$ NB $\angle \ell_1\ell_2 = 60^o$, final rotation 120^o
24	$\ell_2\ell_1n\ell_1\ell_2$	rotates n in plane of ℓ_1, ℓ_2 by twice angle between ℓ_1, ℓ_2 e.g., $(\frac{1}{2}f_1+\frac{\sqrt{3}}{2}f_2)f_1(f_{12}+f_{23}+f_{31})f_1(\frac{1}{2}f_1+\frac{\sqrt{3}}{2}f_2)$ $= (\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12})(f_{12}+f_{23}+f_{31})(\frac{1}{2}+\frac{\sqrt{3}}{2}f_{12})$ $= (\frac{1}{4}+\frac{3}{4})f_{12} + (-\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}) + (\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4})f_{23} + (\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4})f_{31}$ $= f_{12} + (-\frac{1}{2}-\frac{\sqrt{3}}{2})f_{23} + (-\frac{1}{2}+\frac{\sqrt{3}}{2})f_{31}$

3. Practice: building a tetrahedron.

Suppose you want to build a house with a sloping roof. To design the roof trusses you'll need the *dihedral angle*, which is the angle between the planes of the roof and the wall it meets. (Or maybe that angle minus 90 degrees.) If you have dormer windows or additional roof planes, the calculations become more complicated.

We won't build a house but something else which exercises all the same skills needed, and which can easily be checked: a tetrahedron.

We start with an equilateral triangle of unit edges, ℓ_1, ℓ_2 and ℓ_3.

Let's keep the first edge simple: $\ell_1 = f_1$.

Rotate this 60^o (see table, rows 6 and 9). (We don't know ℓ_2 yet but are going to find it out. But we do know the c and s for 60^o.)

$$\begin{aligned}
\ell_2 &= \ell_1(\ell_1\ell_2) \\
&= f_1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2
\end{aligned}$$

And rotate it -60^o.

$$\begin{aligned}
\ell_3 &= \ell_1(\ell_1\ell_3) \\
&= f_1\left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2
\end{aligned}$$

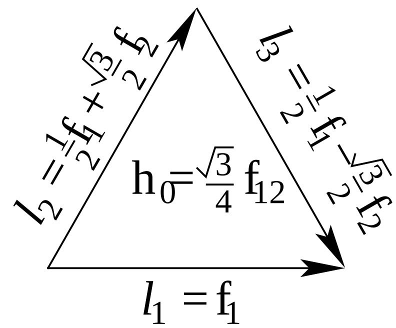
The equilateral triangle with these edges gives one face of the tetrahedron, let's say the base, h_0. Its area is half of the s in $c + sf_{12}$ (see table, row 6).

$$\begin{aligned}
\ell_1\ell_2 &= f_1\left(\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2\right) \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}f_{12} \\
\text{area} &= \frac{1}{2}\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{3}}{4}
\end{aligned}$$

So

$$h_0 = \frac{\sqrt{3}}{4}f_{12}$$

and we have one face of the tetrahedron.



Now we need an edge going up in another dimension to the apex of the tetrahedron. It will have length 1, so normalized, and so we can call it

$$\ell_4 = pf_1 + qf_2 + rf_3$$

with $p^2 + q^2 + r^2 = 1$.

It will be at 60^o to, say, ℓ_1 in some plane n_1. We don't yet know n_1.

(It's best to use a normalized plane. We will easily be able to find the corresponding face because it will have the same area as h_0

$$h_1 = \frac{\sqrt{3}}{4} n_1.)$$

So we know that

$$\begin{aligned} \frac{1}{2} + \frac{\sqrt{3}}{2} n_1 &= \ell_1 \ell_4 \\ &= f_1(pf_1 + qf_2 + rf_3) \\ &= p + qf_{12} - rf_{31} \end{aligned}$$

So $p = 1/2$.

(And $n_1 = \frac{2}{\sqrt{3}}(qf_{12} - rf_{31})$.)

The new edge, ℓ_4, must also be at 60^o to, say, ℓ_2 in another plane n_2, which has a similar relationship to the corresponding face of the tetrahedron

$$h_2 = \frac{\sqrt{3}}{4} n_2.$$

So

$$\begin{aligned} \frac{1}{2} + \frac{\sqrt{3}}{2} n_2 &= \ell_2 \ell_4 \\ &= \left(\frac{1}{2} f_1 + \frac{\sqrt{3}}{2} f_2 \right) (pf_1 + qf_2 + rf_3) \\ &= \frac{p}{2} + \frac{q\sqrt{3}}{2} + \left(\frac{q}{2} - \frac{p\sqrt{3}}{2} \right) f_{12} - r \left(\frac{\sqrt{3}}{2} f_{23} - \frac{1}{2} f_{31} \right) \end{aligned}$$

giving

$$q = \frac{1-p}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

And, from normalization,

$$r^2 = 1 - (p^2 + q^2) = 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3}$$

So $r = \sqrt{\frac{2}{3}}$.

Thus

$$\ell_4 = \frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3$$

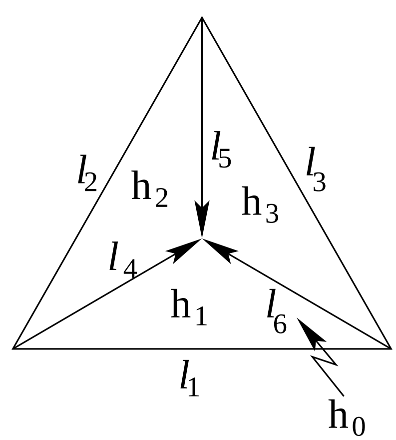
and

$$\begin{aligned} n_1 &= \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{3}}f_{12} - \sqrt{\frac{2}{3}}f_{31}\right) \\ &= \frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31} \\ h_1 &= \frac{\sqrt{3}}{4}n_1 \\ &= \frac{1}{4\sqrt{3}}f_{12} - \frac{1}{\sqrt{6}}f_{31} \end{aligned}$$

We can also find the next face, h_2, from the equation for the second rotation (previous page).

$$\begin{aligned} \frac{\sqrt{3}}{2}n_2 &= \left(\frac{1}{4\sqrt{3}} - \frac{\sqrt{3}}{4}\right)f_{12} + \frac{1}{\sqrt{2}}f_{23} - \frac{1}{\sqrt{6}}f_{31} \\ &= -\frac{1}{2\sqrt{3}}f_{12} + \frac{1}{\sqrt{2}}f_{23} - \frac{1}{\sqrt{6}}f_{31} \\ n_2 &= -\frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} - \frac{\sqrt{2}}{3}f_{31} \\ h_2 &= \frac{\sqrt{3}}{4}n_2 \\ &= -\frac{1}{4\sqrt{3}}f_{12} + \frac{1}{2\sqrt{2}}f_{23} - \frac{1}{2\sqrt{6}}f_{31} \end{aligned}$$

We now have $\ell_1, \ell_2, \ell_3, \ell_4, h_0(n_0), h_1(n_1)$ and $h_2(n_2)$ in the figure.



The rest can be found by rotating.

But we must now be careful. We are working in three dimensions but rotating in two.

For the lines—ℓ_4 must be rotated $\pm 120^o$ in plane f_{12}—we can proceed in either of two ways.

The easiest is to ignore the vertical, f_3, component, and rotate the rest.

Let's try it for $\ell_4 \to \ell_6$. That's $+120^o$.

$$\begin{aligned}
\ell_6 &= \ell_4 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} f_{12}\right) \\
&= \left(\frac{1}{2} f_1 + \frac{1}{2\sqrt{3}} f_2 + \left[\sqrt{\frac{2}{3}} f_3\right]\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} f_{12}\right) \\
&= \left(-\frac{1}{4} - \frac{1}{4}\right) f_1 + \left(-\frac{1}{4\sqrt{3}} + \frac{\sqrt{3}}{4}\right) f_2 + \left[\sqrt{\frac{2}{3}} f_3\right] \\
&= -\frac{1}{2} f_1 + \frac{1}{2\sqrt{3}} f_2 + \sqrt{\frac{2}{3}} f_3
\end{aligned}$$

It's the same as ℓ_4 except for the sign on f_1. That's what the picture expects.

ℓ_5 requires a negative rotation and is different.

$$\begin{aligned}
\ell_5 &= \ell_4 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} f_{12}\right) \\
&= \left(\frac{1}{2} f_1 + \frac{1}{2\sqrt{3}} f_2 + \left[\sqrt{\frac{2}{3}} f_3\right]\right) \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} f_{12}\right) \\
&= \left(-\frac{1}{4} + \frac{1}{4}\right) f_1 + \left(-\frac{1}{4\sqrt{3}} - \frac{\sqrt{3}}{4}\right) f_2 + \left[\sqrt{\frac{2}{3}} f_3\right] \\
&= -\frac{1}{\sqrt{3}} f_2 + \sqrt{\frac{2}{3}} f_3
\end{aligned}$$

It has no f_1 component—again expected—and the same vertical (f_3) component as ℓ_6, and ℓ_4 of course.

We can get away with the above procedure because it is easy to isolate the unchanged component in this case. The proper way to do it is with two inversions (table, row 23).

Let's go for $\ell_4 \to \ell_5$ this time. It's a -120^o rotation so the lines we invert in must be at -60^o to each other.

$$\begin{aligned}
\ell_5 &= \left(\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2\right) f_1 \ell_4 f_1 \left(\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2\right) \\
&= \left(\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2\right) f_1 \left(\frac{1}{2} f_1 + \frac{1}{2\sqrt{3}} f_2 + \sqrt{\frac{2}{3}} f_3\right) f_1 \left(\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2\right) \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2} f_{12}\right) \left(\frac{1}{2} f_1 + \frac{1}{2\sqrt{3}} f_2 + \sqrt{\frac{2}{3}} f_3\right) \left(\frac{1}{2} - \frac{\sqrt{3}}{2} f_{12}\right) \\
&= \left(\frac{1}{8} - \frac{3}{8} + \frac{1}{8} + \frac{1}{8}\right) f_1 + \left(\frac{1}{8\sqrt{3}} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}\right) f_2 \\
&\quad + \left(\frac{1}{2\sqrt{6}} + \frac{\sqrt{3}}{2\sqrt{2}}\right) f_3 + \left(\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) f_{123}
\end{aligned}$$

$$= -\frac{1}{\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3$$

which we got before.

We can now save a little algebra by doing $\ell_5 \to \ell_6$.

$$\begin{aligned}
\ell_6 &= \left(\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2\right) f_1\ell_5 f_1 \left(\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2\right) \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right)\left(-\frac{1}{\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \left(-\frac{1}{4} - \frac{1}{4}\right) f_1 + \left(-\frac{1}{4\sqrt{3}} + \frac{\sqrt{3}}{4}\right) f_2 + \left(\frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2\sqrt{2}}\right) f_3 + \left(\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) f_{123} \\
&= -\frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3
\end{aligned}$$

as before.

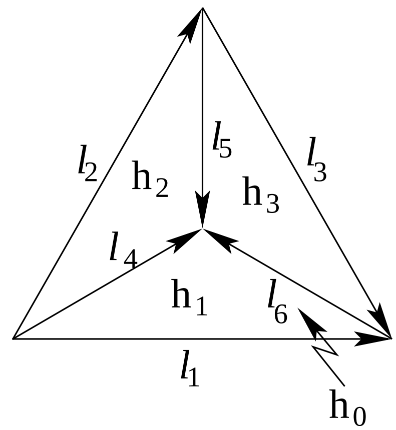
To rotate plane n_1 to n_2 and n_3 we'll use inversions of $\pm 60^o$ in plane $n_0 = f_{12}$. Just as ℓ_2 is -120^o from ℓ_1, so h_2 is -120^o from h_1.

$$\begin{aligned}
n_2 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right) n_1 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right)\left(\frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \left(\frac{1}{12} + \frac{1}{4}\right) f_{12} + \left(-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\right) f_{23} + \left(-\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}}\right) f_{31} + \left(\frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}}\right) \\
&= \frac{1}{3}f_{12} - \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}
\end{aligned}$$

And a positive rotation for n_3. (It's the same algebra with a few sign changes.)

$$\begin{aligned}
n_3 &= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right) n_1 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}f_{12}\right)\left(\frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}f_{12}\right) \\
&= \left(\frac{1}{12} + \frac{1}{4}\right) f_{12} + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right) f_{23} + \left(-\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}}\right) f_{31} + \left(-\frac{1}{4\sqrt{3}} + \frac{1}{4\sqrt{3}}\right) \\
&= \frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}
\end{aligned}$$

In summary



$$
\begin{aligned}
\ell_1 &= f_1 \\
\ell_2 &= \frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2 \\
\ell_3 &= \frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2 \\
\ell_4 &= \frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3 \\
\ell_5 &= -\frac{1}{\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3 \\
\ell_6 &= -\frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3 \\
h_0 &= \frac{\sqrt{3}}{4}f_{12} \\
h_1 &= \frac{\sqrt{3}}{4}n_1 = \frac{1}{4\sqrt{3}}f_{12} - \frac{1}{\sqrt{6}}f_{31} \\
h_2 &= \frac{\sqrt{3}}{4}n_2 = -\frac{1}{4\sqrt{3}}f_{12} + \frac{1}{2\sqrt{2}}f_{23} - \frac{1}{2\sqrt{6}}f_{31} \\
h_3 &= \frac{\sqrt{3}}{4}n_2 = \frac{1}{4\sqrt{3}}f_{12} + \frac{1}{2\sqrt{2}}f_{23} + \frac{1}{2\sqrt{6}}f_{31}
\end{aligned}
$$

4. Visualizing planar components.

It is easy to visualize lines and their components, For example

$$\ell_4 = \frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3$$

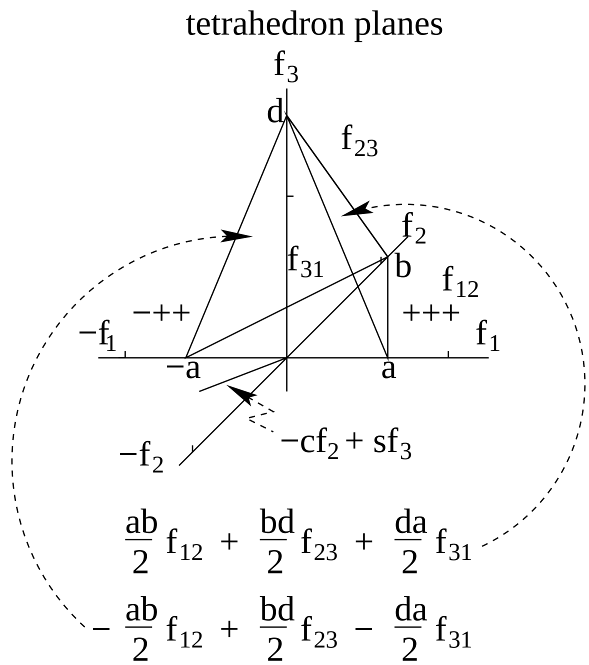
has f_1-component $\frac{1}{2} = 0.5$, f_2-component $\frac{1}{2\sqrt{3}} \approx 0.29$ and f_3-component $\sqrt{\frac{2}{3}} \approx 0.82$.

These are all positive, so as ℓ_4 moves up 0.5 in the f_1 direction it also moves up 0.29 in the f_2 direction and up 0.82 in the f_3 direction (out of the page).

A negative value for any of these three components would force the line “down” in that direction.

We can also see components of a plane as the projections of the area of the plane on the basis planes f_{12}, f_{23} and f_{31}

Here are the planes n_3 (on the right) and n_2 (on the left) of the tetrahedron. I've rotated f_1 and f_2 to align with the tetrahedron in earlier figures.



I've used positive numbers, a, b and d, for the intersections of the planes on the axes, and given the expressions for the planes in terms of these.

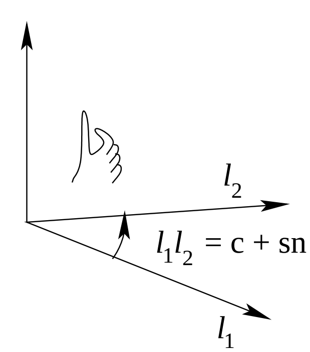
So n_3 has three positive coefficients and creates the triangle shown in the $+++$ octant.

n_2 as shown here has two negative coefficients and appears in the $-++$ octant. This is indeed the form given for h_2 in the summary at the end of the previous section. But it is the negative of the form given for n_2 by rotation the page before that.

Planes and faces also have directions, and their expression can change sign depending on which side of the plane we're looking from. Different methods of deriving the planar expression can give different signs.

So I've shown planes in directions that give an even number of $-$ signs, 0 or 2. Their negations are the same planes viewed from the other side.

We can find the direction of a plane by constructing it as the product of two normalized lines in it—not collinear—together with the "right-hand rule".



We have ℓ_1 and ℓ_2 in a plane n. The direction of the plane is the direction of the thumb of your right hand when you've curled the fingers to point in the direction of the arc-arrow in the angle from ℓ_1 to ℓ_2.

We can construct the planes of the tetrahedron with their directions *outwards* from the tetrahedron in the summary figure above.

$$\begin{aligned} n_0 : \ell_2\ell_1 &= \left(\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2\right) f_1 \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}f_{12} \\ n_0 &= -f_{12} \end{aligned}$$

with n_0 normalized, $n_0n_0 = -1$. Here $c = \frac{1}{2}$ and $s = \frac{\sqrt{3}}{2}$: the lines ℓ_2 and ℓ_1 are 60^o apart..

$$\begin{aligned} n_1 : \ell_1\ell_4 &= f_1 \left(\frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3\right) \\ &= \frac{1}{2} + \frac{1}{2\sqrt{3}}f_{12} - \sqrt{\frac{2}{3}}f_{31} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}n_1 \\ n_1 &= \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{3}}f_{12} - \sqrt{\frac{2}{3}}f_{31}\right) \\ &= \frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31} \end{aligned}$$

normalizing n_1.

$$\begin{aligned} n_2 : \ell_4\ell_2 &= \left(\frac{1}{2}f_1 + \frac{1}{2\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3\right)\left(\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2\right) \\ &= \frac{1}{4} + \frac{1}{4} + \left(\frac{\sqrt{3}}{4} - \frac{1}{4\sqrt{3}}\right) f_{12} - \frac{1}{\sqrt{2}}f_{23} + \frac{1}{\sqrt{6}}f_{31} \\ &= \frac{1}{2} + \frac{1}{2\sqrt{3}}f_{12} - \frac{1}{\sqrt{2}}f_{23} + \frac{1}{\sqrt{6}}f_{31} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}n_2 \\ n_2 &= \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{3}}f_{12} - \frac{1}{\sqrt{2}}f_{23} + \frac{1}{\sqrt{6}}f_{31}\right) \\ &= \frac{1}{3}f_{12} - \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31} \end{aligned}$$

$$\begin{aligned} n_3 : \ell_5\ell_3 &= \left(-\frac{1}{\sqrt{3}}f_2 + \sqrt{\frac{2}{3}}f_3\right)\left(\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2\right) \\ &= \frac{1}{2} + \frac{1}{2\sqrt{3}}f_{12} + \frac{1}{\sqrt{2}}f_{23} + \frac{1}{\sqrt{6}}f_{31} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \end{aligned}$$

$$n_3 = \frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}$$

You can compare the signs with our previous derivations of these planes, and so determne which side we are looking from.

I did not show plane

$$n_1 = \frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}$$

in the diagram of tetrahedral planes.

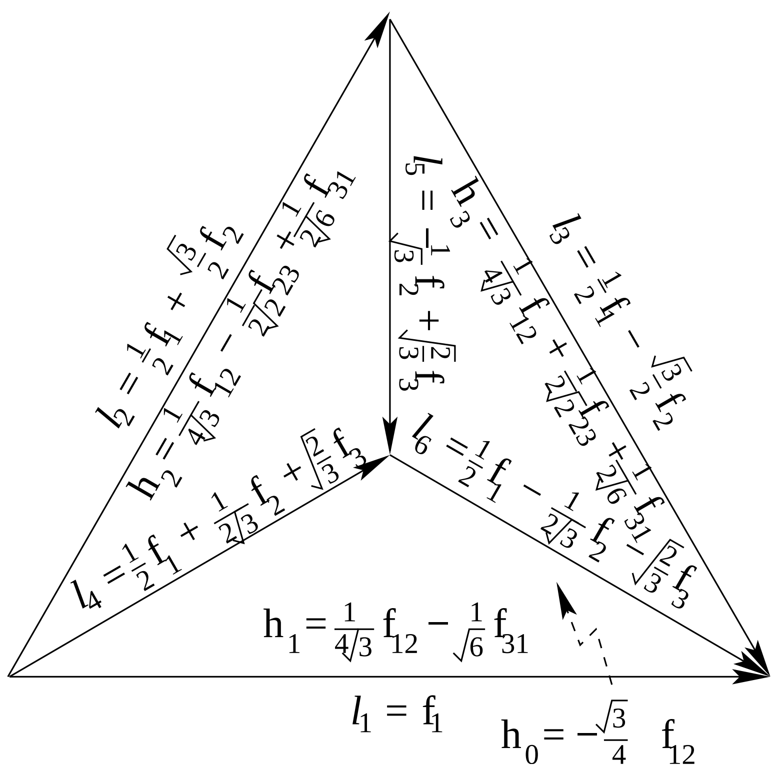
That is because one of the edges is parallel to f_1: one of the triangular components would have to be zero, and the others could not be triangles.

What I did show was the line *perpendicular* to that plane

$$\begin{aligned} -n_1 f_{123} &= -\left(\frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}\right) f_{123} \\ &= \frac{1}{3}f_3 - \frac{2\sqrt{2}}{3}f_2 \end{aligned}$$

which should now make it easy to see how the plane goes.

Here is the tetrahedron we have. Edges are directed as shown. Faces are directed as seen from outside the tetrahedron.



5. Dihedral angles.

Just as the product of two (normalized) lines gives the angle between them, so the product of two (normalized) planes gives the *dihedral* angle between the planes.

So we can find the angles between pairs of planes in the tetrahedron. By symmetry, all six of them are the same.

$$\begin{aligned} n_0 n_1 &= -f_{12}\left(\frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}\right) \\ &= \frac{1}{3} + \frac{2\sqrt{2}}{3}f_{23} \end{aligned}$$

in which $c = \frac{1}{3}$ and $s = \frac{2\sqrt{2}}{3}$ correspond to an angle of about 70.5^o in the normalized plane f_{23}. You should be able to see from the picture of the tetrahedron that this is the correct plane for that dihedral angle to be in.

Similarly

$$\begin{aligned} n_0 n_2 &= -f_{12}\left(\frac{1}{3}f_{12} - \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \\ &= \frac{1}{3} - \frac{\sqrt{2}}{3}f_{23} - \sqrt{\frac{2}{3}}f_{31} \\ &= \frac{1}{3} - \frac{2\sqrt{2}}{3}\frac{3}{2\sqrt{2}}\left(\frac{\sqrt{2}}{3}f_{23} - \sqrt{\frac{2}{3}}f_{31}\right) \\ &= \frac{1}{3} - \frac{2\sqrt{2}}{3}\left(\frac{1}{2}f_{23} + \frac{\sqrt{3}}{2}f_{31}\right) \end{aligned}$$

This gives an angle of about -70.5^o in the normalized plane $\frac{1}{2}f_{23} + \frac{\sqrt{3}}{2}f_{31}$.

Changing a sign gives

$$\begin{aligned} n_0 n_3 &= -f_{12}\left(\frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \\ &= \frac{1}{3} - \frac{2\sqrt{2}}{3}\left(\frac{1}{2}f_{23} - \frac{\sqrt{3}}{2}f_{31}\right) \end{aligned}$$

Same angle, slightly different plane. It is harder to visulalize the two dihedral planes we just got for $n_0 n_2$ and $n_0 n_3$ but we can check the angle between *them*.

$$\begin{aligned} \left(\frac{1}{2}f_{23} + \frac{\sqrt{3}}{2}f_{31}\right)\left(\frac{1}{2}f_{23} - \frac{\sqrt{3}}{2}f_{31}\right) &= -\frac{1}{4} + \frac{3}{4} + \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)f_{12} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}f_{12} \end{aligned}$$

which is 60^o in the f_{12} plane, as a little introspection says it should be.

Combining n_1, n_2 and n_3 is a little more complicated but reveals the same 70.5^o each time.

$$\begin{aligned} n_1 n_2 &= \left(\frac{1}{3} - \frac{2\sqrt{2}}{3}f_{31}\right)\left(\frac{1}{3}f_{12} - \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \\ &= -\frac{1}{9} + \frac{4}{9} + \frac{4}{3\sqrt{3}}f_{12} + \left(\frac{2\sqrt{2}}{9} + \frac{\sqrt{2}}{9}\right)f_{23} + \frac{\sqrt{2}}{3\sqrt{3}}f_{31} \\ &= \frac{1}{3} + \frac{4}{3\sqrt{3}}f_{12} + \frac{\sqrt{2}}{3}f_{23} + \frac{\sqrt{2}}{3\sqrt{3}}f_{31} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3} + \frac{2\sqrt{2}}{3}\frac{3}{2\sqrt{2}}\left(\frac{4}{3\sqrt{3}}f_{12} + \frac{\sqrt{2}}{3}f_{23} + \frac{\sqrt{2}}{3\sqrt{3}}f_{31}\right) \\
&= \frac{1}{3} + \frac{2\sqrt{2}}{3}\left(\sqrt{\frac{2}{3}}f_{12} + \frac{1}{2}f_{23} + \frac{1}{2\sqrt{3}}f_{31}\right)
\end{aligned}
$$

And, changing a sign,

$$
\begin{aligned}
n_1n_3 &= \left(\frac{1}{3} - \frac{2\sqrt{2}}{3}f_{31}\right)\left(\frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \\
&= \frac{1}{3} - \frac{2\sqrt{2}}{3}\left(\sqrt{\frac{2}{3}}f_{12} - \frac{1}{2}f_{23} + \frac{1}{2\sqrt{3}}f_{31}\right)
\end{aligned}
$$

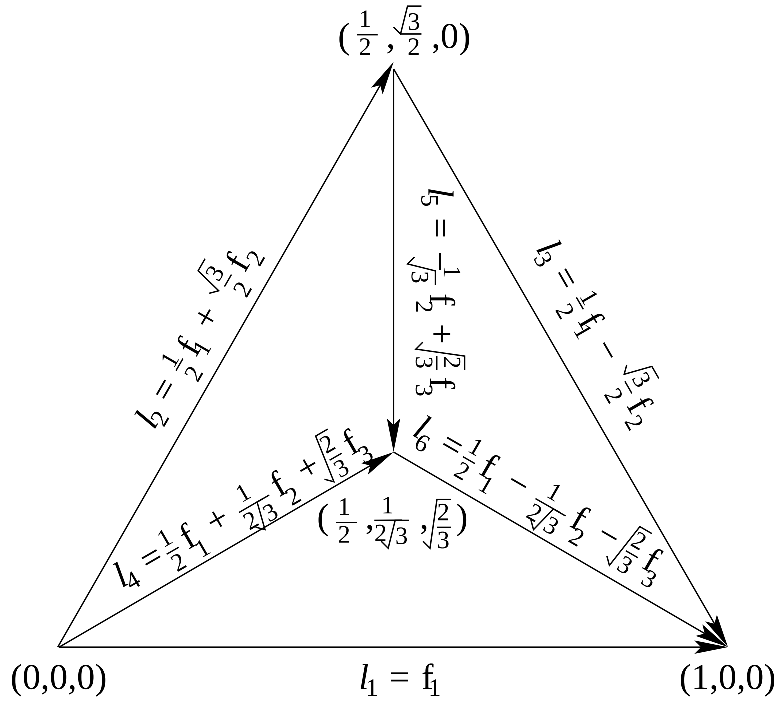
The dihedral planes for these two meet at a 60^o angle in the plane $\frac{1}{3}f_{12} - \frac{2\sqrt{2}}{3}f_{31}$.
Finally,

$$
\begin{aligned}
n_2n_3 &= \left(\frac{1}{3}f_{12} - \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right)\left(\frac{1}{3}f_{12} + \sqrt{\frac{2}{3}}f_{23} + \frac{\sqrt{2}}{3}f_{31}\right) \\
&= \frac{1}{9} - \frac{2}{3} + \frac{2}{9} + \left(\frac{2}{3\sqrt{3}} + \frac{2}{3\sqrt{3}}\right)f_{12} + \left(\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9}\right)f_{23} + \left(-\frac{\sqrt{2}}{3\sqrt{3}} - \frac{\sqrt{2}}{3\sqrt{3}}\right)f_{31} \\
&= -\frac{1}{3} + \frac{4}{3\sqrt{3}}f_{12} - \frac{2\sqrt{2}}{3\sqrt{3}}f_{31} \\
&= -\frac{1}{3} + \frac{2\sqrt{2}}{3}\frac{3}{2\sqrt{2}}\left(\frac{4}{3\sqrt{3}}f_{12} - \frac{2\sqrt{2}}{3\sqrt{3}}f_{31}\right) \\
&= -\frac{1}{3} + \frac{2\sqrt{2}}{3}\left(\sqrt{\frac{2}{3}}f_{12} - \frac{1}{\sqrt{3}}f_{31}\right)
\end{aligned}
$$

6. Absolute coordinates..

The elements of the reflection algebra are all *intervals* and so one symbol, f_1 or f_{12} and so on, denotes an element which may be placed anywhere in the space.

But we can anchor them by going outside of the reflection algebra and specifying *coordinates* of the end points of lines.



I've given the origin arbitrarily as the lowest left-hand vertex.You can see how the direct edges from there, ℓ_1, ℓ_2 and ℓ_4, give the coordinates of the other vertices. (For convenience I've reversed the direction of ℓ_6 from previous diagrams.)

You can also see how edges can be added or subtracted to give other edges.

$$\begin{aligned}
\ell_2 + \ell_3 &= \ell_1 \\
\ell_4 + \ell_6 &= \ell_1 \\
\ell_2 + \ell_5 &= \ell_4 \\
\ell_5 + \ell_6 &= \ell_3 \\
\ell_4 - \ell_5 &= \ell_2
\end{aligned}$$

and so on.