Excursions in Computing Science: Week 3. Speed of Light

T. H. Merrett*
McGill University, Montreal, Canada

May 24, 2022

I. Prefatory Notes

1. How far is it from your home to class?

2. How can we use *time* to answer a *distance* query?

We presume a *speed*: walking speed, bus speed, metro speed, .., light speed

Speed mixes space and time:

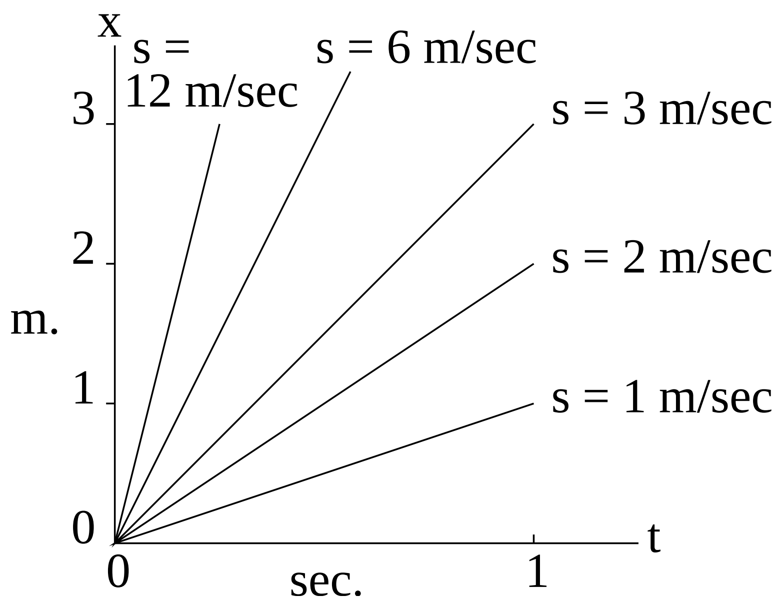
We need an operator on

$$\begin{pmatrix} t \\ x \end{pmatrix}$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = S \begin{pmatrix} t \\ x \end{pmatrix}$$

3. Here is one way of thinking of space, time and speed.

 The author gratefully acknowledges support from the taxpayers of Québec and of Canada who have paid his salary and research grants while this work was developed at McGill University, and from his students and their funding agencies.



s is speed. (Don't mix it up with $s = \sin()$!) So we'll call it v for *velocity* from now on. (Which is a better word, since it also includes *direction*, but that's not going to concern us yet.)

Galileo and Newton considered time to be absolute, i.e., unaffected by speed:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

The v term turns time into space, allowing us to live five minutes from class.

Woops! Why the $-$ sign?

Because t' and x' are intended to be the time and space experienced by the the observer who has been transformed by moving at speed v.

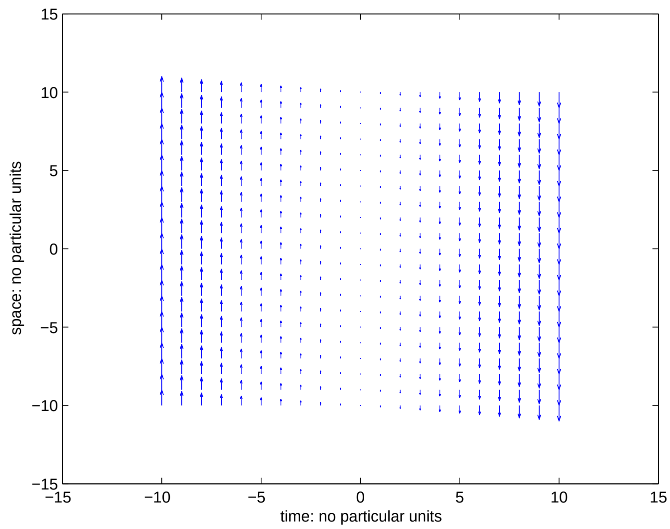
If I am moving past you at speed 1 m/sec, starting at your position at time $t = 0$, then something 3 m ahead of you is, for me:

- 3 m ahead at $t = 0$;
- 2 m ahead at $t = 1$;
- 1 m ahead at $t = 2$;
- 0 m ahead at $t = 3$;
- 1 m behind at $t = 4$;
- 2 m behind at $t = 5$;

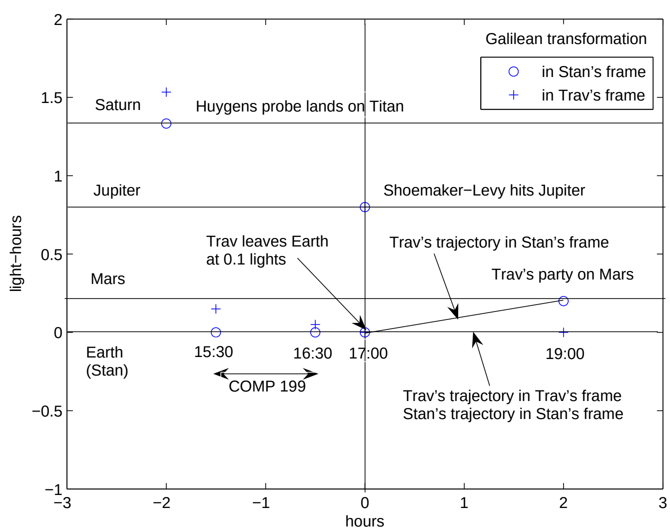
$$\begin{aligned} \begin{pmatrix} 0 \\ 3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &\vdots \\ \begin{pmatrix} 5 \\ -2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \end{aligned}$$

Note that time is always the same for both of us.

Here is what the Galilean velocity transformation does to time and space.



To picture what this means, here are the time and space coordinates of a set of events as perceived by Stanley (who is *stan*ding still on earth) and Travis (who is *trav*elling at $v = 0.1$, leaving Earth at 17:00 to catch a party on Mars at 19:00). Also shown are comet Shoemaker-Levy colliding with Jupiter (this actually happened over the period 16–22 July 1994, but we're pretending it just happened at 17:00), the Huygens probe, from Cassini-Huygens, landing on Saturn's moon Titan (which also did not just happen at 15:00, but landed on Christmas Day 2004), and, back on Earth, the start and end of a COMP 199 class.



4. If I am travelling at v relative to you and my daughter is travelling at u relative to me, then relative to you she is moving at $u + v$:

$$\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u-v & 1 \end{pmatrix}$$

So if I have a long enough line of descendents, each moving a finite amount faster than heir parent, we could build up unlimited speed relative to you.

Now I'd like to say that Nature Abhors Infinities.

So there must be a maximum speed.

Call it c. (Don't mix this up with cos()! I won't change notation this time.)

We can see that now speed arithmetic must be very weird, because $u+v$ must always be less than c, for two speeds u and v. Say, $u = 3c/4$ and $v = 2c/3$.

So if I am travelling at $3c/4$ relative to you and a throw a ball at $3c/4$ relative to me, the ball can *not* be going at $3c/2$ relative to you.

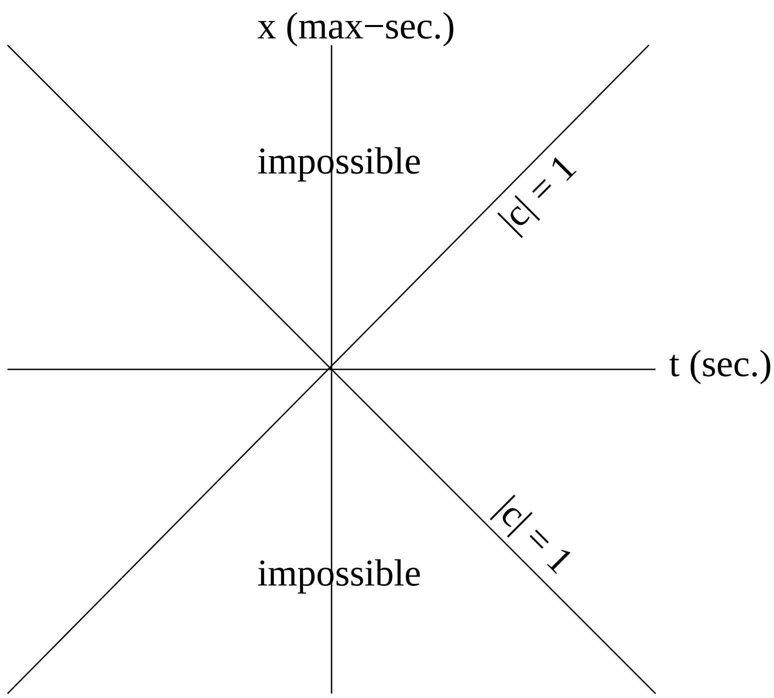
c must be very big, because we don't experience this every day. (Or maybe I am wrong and Nature doesn't mind infinities.)

5. Let's see if we can invent an operator, S, which increases speed as

$$G \quad \textbf{defined as} \quad \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

does, but, unlike G, does not allow it to exceed c (either forwards or backwards).

To make calculations easy, we'll *redefine* the space unit to be $c \times 1$ second. We'll call this unit a "max-second". So when we show t in seconds and x in max-seconds, c will be 1.



The $\mid c \mid= 1$ lines are the "fixed points" of the S operator. (It would be better to call them the "fixed directions" of S.)

$$\begin{aligned} S\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ S\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

The vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are the directions of the c lines, and choosing $c = 1$ has the handy benefit that they are "orthogonal" (a mathematician's word for "at right angles", but one so much used we should introduce it):

$$0 = (1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \cos\left(\text{angle}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right)$$

λ_1 and λ_2 are just numbers, called "eigenvalues". These say that, if the velocity ever reaches c, further velocity transformations don't change the direction of the c-line. It may move the points on that line further out or in, depending on whether the eigenvalues are big or small; since only the direction of the line gives the velocity, this doesn't matter.

6. Let's try

$$\begin{aligned}
S &= \frac{\lambda_1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1,1) + \frac{\lambda_2}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1,-1) \\
&= \frac{\lambda_1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\lambda_2}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ b & a \end{pmatrix}
\end{aligned}$$

I've simplified the λs down to a and b, but we can get them back again:

$$\begin{aligned}
\lambda_1 &= a + b \\
\lambda_2 &= a - b
\end{aligned}$$

Check that this S does leave the c-lines fixed, as required!

7. So we know that S is built on two numbers, a on the diagonal and b off the diagonal (and that it is symmetric), but we don't know how those numbers relate to v.

The Galilean transform,

$$G = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

seems to be correct for everyday life, i.e., for low speeds given by small v. So let's try

$$S = \gamma\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

This is symmetrical and based on two numbers. One is v, making S look something like G. The other is γ, another number.

To figure out what γ is, let's insist that the "determinant" of S is 1. I'll try to justify this shortly.

$$1 = \begin{vmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{vmatrix} = \gamma^2(1 - v^2)$$

So if I can convince you of that, then

$$\gamma = 1/\sqrt{1-v^2}$$

It should help a little (but not completely) that $\gamma \to 1$ as v gets small. And remember that, in normal units (not max-seconds), v becomes v/c, which is extremely small in daily life.

What happens to the upper v in S when we change to normal units? To allow G to approximate S in everyday life, it must become *very* small, much smaller than the lower v, to give G that upper 0.

This upper v mixes space into time, and does not agree with our low-speed intuitions at all. The lower v mixes time into space, and that is no problem: it's just what we are used to velocity doing.

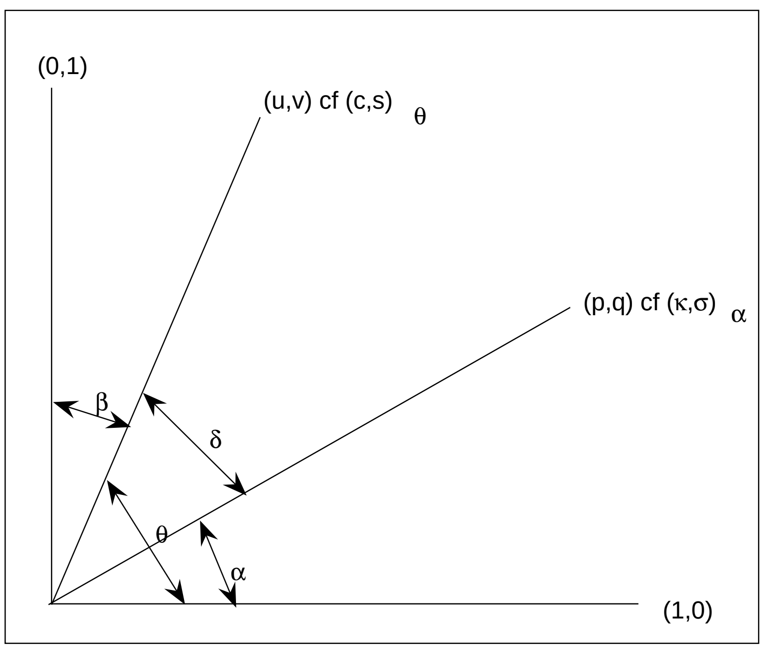
8. **Digression on determinants.**

What does the determinant mean, and why should it be 1 in S?

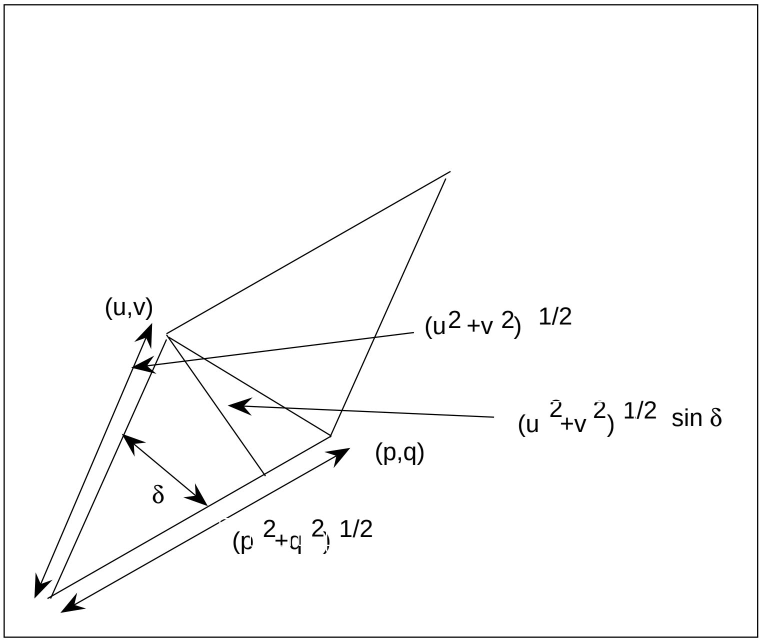
(This is not a physical argument, but it is a shortcut to the right theory. All theories must be tested by *observation* and *experiment*, so it doesn't really matter how we get to the theory. It's nice to have simple ideas, which is what I am trying to do: we think more clearly with them. But simple ideas can be shot down by a measurement just as fast as complicated ideas—maybe faster—or beautiful ideas or logical ideas.)

Any matrix

$$\begin{pmatrix} p & u \\ q & v \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$$



$$\begin{aligned} \kappa &= p/\sqrt{p^2+q^2} \\ \sigma &= q/\sqrt{p^2+q^2} \\ c &= u/\sqrt{u^2+v^2} \\ s &= v/\sqrt{u^2+v^2} \\ \sin\delta &= \sin(\theta-\alpha) = \kappa s - \sigma c \end{aligned}$$



area of parallelepiped =

$$\sqrt{u^2+v^2}\sin\delta\sqrt{p^2+q^2} \quad = \quad pv - qu = \begin{vmatrix} p & u \\ q & v \end{vmatrix}$$

So if $\det S > 1$ then transformation increases area, and repeated transformations give an unbounded area.

And if $\det S < 1$ then transformation decreases area, and repeated transformations give an unboundedly small area. ("Nature Abhors a Vacuum" follows from "Nature Abhors Infinities" if we include the infinitely small.)

Only $\det S = 1$ is stable.

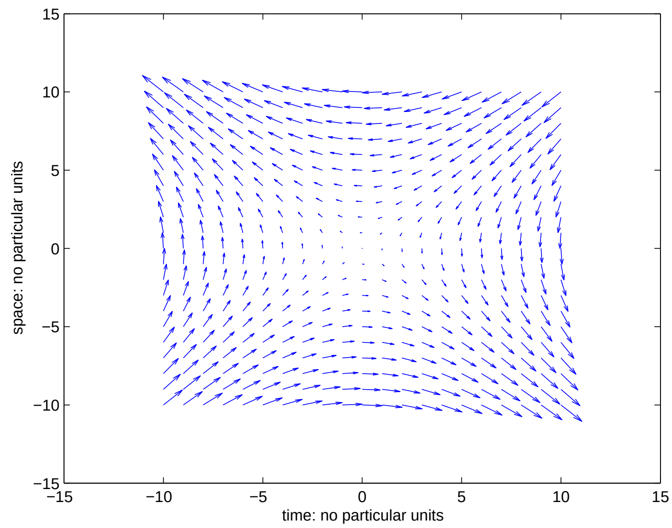
Compare determinants for rotation, R, and speed transformation, S.

$$\det R \quad = \quad \begin{vmatrix} c & -s \\ s & c \end{vmatrix} = c^2 + s^2 = 1$$
$$\det S \quad = \quad \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2 = 1$$

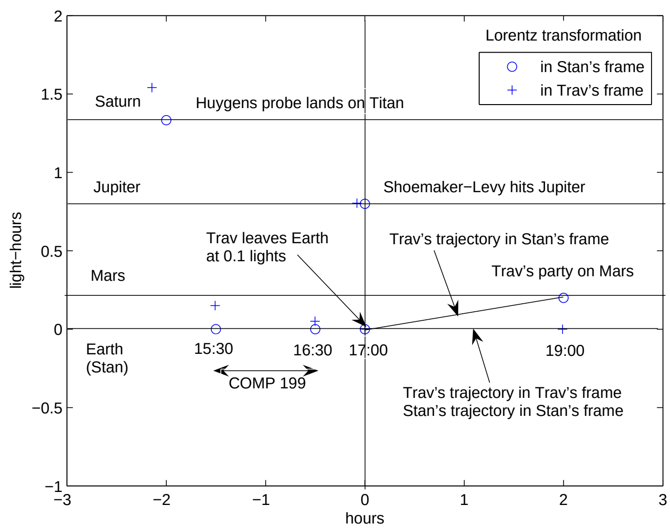
9. We call the special form of S we've derived, the Lorentz transformation.

$$S \quad = \quad \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

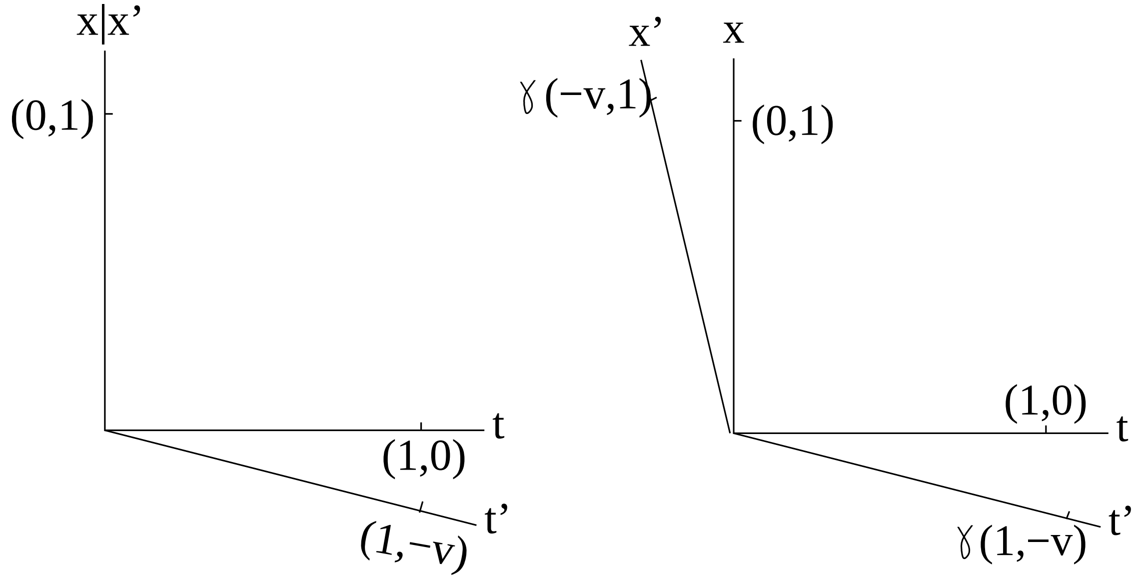
It is a shear transformation.



Here are Stanley and Travis again, with Travis' perception derived from the Lorentz transformation. Note that Stanley's perceptions are unchanged from before (Note 3), but Travis' are subtly shifted, notably in the time coordinates, from what Galileo did to them.



Both Galileo and Lorentz give new time axes to the traveller



Galileo (G) Lorentz (S)

but Lorentz also gives a new space axis.

Since Lorentz mixes time with space and space with time, we should really not talk about time and space separately, but as a single entity, "timespace".

N.B. as desired

$$\begin{aligned} G(v)G(u) &= G(u+v) \\ S(v)S(u) &\neq S(u+v) \end{aligned}$$

Check it!

10. **Using the Lorentz transformation.**

What are (t,x) and (t',x') in

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

Suppose that you and I are at the origin, $(t,x)=(0,0)$, but you are moving at velocity v (in the x direction) relative to me. Then, any event that I observe at (t,x) in my coordinate system, you will observe at (t',x') in your coordinate system, according to the Lorentz transformation. What is remarkable and counter to everyday experience, is that how we perceive *time* depends on our velocity. We are used to *space* doing this, but not time.

How are time and space affected by the Lorentz transformation? If my event, $(t,x)=(5,4)$, and your velocity is 3/5, show that your event (same event), $(t',x')=(13/4,5/4)$. So, for you relative to me, time dilated from 5 to 3.25, and distance contracted from 4 to 1.25.

Suppose you are again going past me with velocity 3/5, that we synchronize our watches as you pass me at (0,0), and that 5 seconds later (my time) I read your clock. This event will be at $(t,x)=(5,3)$ for me, because after $t=5$ seconds, you will be at $x=3$ units as far as I am concerned. Show that your clock says only 4 seconds have elapsed. Show that the ratio, 5/4, equals $\gamma = 1/\sqrt{1-v^2}$. Note also that the transformation confirms that you stayed perfectly still, at your space-origin $x' = 0$, during this interval.

For a third time, you are passing me with velocity 3/5. The moment the back of your vehicle passes me at the origin you switch on your lights, back and front. I see your back light come on right away, but I only see your front light come on, 5 units away, 3 seconds later. Show that you measure your vehicle as 4 units long (and the lights come on simultaneously for you). Note again that the ratio, 5/4, equals γ.

11. **Physical principles behind special relativity.**

- Laws of physics are unaffected by uniform speed (Galileo).
- Speed of light is a law of physics (Maxwell).
- So lightspeed is the maximum
- and a meter is *defined* as 1/299,972,458 max-sec. ("light-sec.").

Lightspeed is $\sim$ 0.3 Gm/sec.

12. **Summary**

(These notes show the trees. Try to see the forest!)

- Fixed-point vector of a transformation ("eigenvector").
- Related terminology: "eigenvalue", "orthogonal", "normal", "orthonormal".
- Shear transformations: Galilean, Lorentz.
- Nature abhors infinities ... lightspeed is a law of nature.
- It doesn't matter how we arrive at a scientific theory.

A five-step derivation of special relativity.

1. Show that for the primed observer moving at velocity v, $x' = x - vt$.
2. Put this into a matrix to get the Galilean transformation.
3. Symmetrize the matrix so that the fastest possible velocity is 1.
4. Make the determinant 1 to give the divisor $\sqrt{1 + v^2}$ and the Lorentz transformation.
5. Because time must be distance/velocity, show that the v term in the result for t' must really be v/c^2 where c is the fastest possible velocity, lightspeed.

Note that the negative signs mean that the shear transformation spreads the t and x axes *further* apart than the original right angle, and that the unit determinant, i.e., the t-x area stays the same under the transformation, thus means that both t' and x' must be smaller then t and x respectively. That is, time dilates and space contracts.

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. To underscore that the transformations we're discussing this Week are simply those relating the measurements made by two observers, compare the timespace measurements of observers Stan and Trav moving at different speeds

$$\underbrace{\begin{pmatrix} t' \\ x' \end{pmatrix}}_{\text{Trav}} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \underbrace{\begin{pmatrix} t \\ x \end{pmatrix}}_{\text{Stan}} \quad \text{or} \quad \underbrace{\begin{pmatrix} t' \\ x' \end{pmatrix}}_{\text{Trav}} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \underbrace{\begin{pmatrix} t \\ x \end{pmatrix}}_{\text{Stan}}$$

(this applies to either Galilean or Lorentz boost) with the spacespace measurements of observers Ann and Bob using coordinate systems rotated with respect to each other

$$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{\text{Bob}} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{Ann}}$$

(and why did I change the sign of s from what we were using in Excursion *Cosine and sine* of Week iv?).

2. Here is another way to see that $x' = x - vt$, i.e., that there is a negative sign on v in the Galilean transformation. Suppose Stan and Trav are both at $x = 0$ and are both looking at a cookie at $x = 10$ meters. They can both walk to the cookie at a leisurely 1 m/s. So Stan needs 10 seconds to get to the cookie, and we can take that as the (time-based) measurement of Stan's distance x from the cookie. (We're actually pretending lightspeed is 1 m/s here because lightspeed is what we should use for time-based measurements of distance.)
Trav, though, is on a slow train moving at $v = 0.1$ m/s towards the cookie. So Trav can walk to the cookie in $10/(1+0.1) \approx 9$ seconds. That is Trav's (time-based) measure of distance, x'. So x' is shorter than x, hence the subtraction.
For $v \ll c$ show that this argument gives $x' = x - vt$.

3. The Lorentz transformation (named in Note 9) is also called the *boost* transformation. This can be misleading because "boost" would seem to mean increasing your own velocity. How would you change the Lorentz transformation so that it could be more convincingly interpreted as a "boost"? Hint: think about rotations (Note 8 of Week iv) and the difference between a rotation and *rotating the coordinate axes.*

4. If we push on one end of a long rod which is free to move, the other end then moves. Does this violate the statement that nothing moves faster than light?

5. Standing at a very long breakwater wall on the ocean coast, we see a wave arrive at an extremely small angle to the wall, so small that the arrival of the wave at one end occurs sooner after its arrival at the other end than light would take to travel between the two ends. Does this violate the lightspeed limit?

6. "The fourth dimension is time." If so, what is "the third dimension"?

7. To draw a time-space diagram with c, the speed of light, shown as 1, what must the units on the space axis be? Alternatively, what must the units on the time axis be?

8. Taking the speed of light to be 300 megameters/sec., find out how long it takes light to cross a continent of 6000 km, to cross a room, to cross a 6 mm integrated circuit chip. Find out how far a light second is, a light year, how wide a galaxy of 100,000 light years is. Do these calculations in your head (yes, even turning years into seconds) and give the results in seconds and meters, prefixed by the appropriate one of: femto, pico, nano, micro, milli, —, kilo, mega, giga, tera, peta, exa Do not calculate more than two significant figures, but try to give results correct to within 10%
P.S. How long is a kilosecond? a megasecond? a gigasecond?

9. Use MATLAB to apply the Galilean transformation

$$\begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

repeatedly, and confirm that the resulting speed never stops growing (or shrinking).

10. Multiply matrix S (Note 6) by vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and show that these vectors are fixed points of S and λ_1 and λ_2 are eigenvalues of S.

11. If $\vec{u}$ and $\vec{v}$ are "fixed-point" vectors of a transformation T, why must they be "orthogonal" and "normalized" (i.e., "orthonormal") to derive

$$T = \lambda_1 \vec{u}^T \vec{u} + \lambda_2 \vec{v}^T \vec{v}$$

where $T\vec{u} = \lambda_1 \vec{u}$ and $T\vec{v} = \lambda_2 \vec{v}$?

12. What is (are) the fixed point(s) of the Galilean transformation?

13. Use MATLAB to apply the Lorentz transformation

$$\frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

repeatedly, with $v < 1$, and confirm that the resulting speed never exceeds 1.

14. Show that

$$\frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \frac{1}{\sqrt{1-u^2}} \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix} = \frac{1}{\sqrt{1-w^2}} \begin{pmatrix} 1 & -w \\ -w & 1 \end{pmatrix}$$

where $w = (u+v)/(1+uv)$. If u and v are both 3/5, what is w?

15. Find the matrices which
a) leave fixed the vectors $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}/2$ and $\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}/2$
b) map $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}/2$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}/2$
What are the determinants of these matrices?

16. Use the MATLAB `abs()` function to find the above determinants.

17. As an alternative to requiring some abstract "area" (the determinant of the transform matrix) to be 1 in order to derive $\gamma = 1/\sqrt{1-v^2}$, try arguing that the inverse of the transform

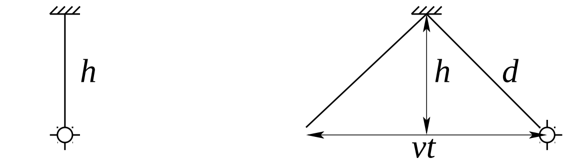
$$\gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

must be

$$\gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$$

(Why?)

18. Here is an argument giving time dilation. It assumes lightspeed, c, is the same in both moving (Trav) and stationary (Stan) frames, and that h, the distance perpendicular to the motion, is unaffected. Trav shines light vertically upward at a mirror and measures (t') how long it takes to go both ways.



ct' = 2h $\qquad$ $d^2 = h^2 + (vt/2)^2$

Trav's view $\qquad$ Stan's view

Stan sees the light travel upwards then downwards on diagonals, taking time t where $c^2t^2 = 2^2d^2 = 2^2h^2 + v^2t^2$. Show that $t = t'/\sqrt{1-(v/c)^2}$.
Reverse the argument and show that $t' = t/\sqrt{1-(v/c)^2}$. What's wrong? Hint: t and t' are time *intervals*; what does the full Lorentz transformation say?

19. **Lorentz ($c \neq 1$).** Give an argument to show that, if $c \neq 1$, the Lorentz transformation should be written
$$\gamma \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix}$$
How similar is *this* to the Galilean transformation? What is γ in these units?

20. **The weird fizzmezh matrices.** When changing physical measures ("fizzmezh") from units in which lightspeed equals 1, e.g., time in seconds, space in light-seconds, we must change v to v/c and in the timespace vector we must change x to x/c, from light-seconds to, say, meters:
$$\begin{pmatrix} 1 & \\ & 1/c \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$
Using this matrix and its inverse, calculate
$$\begin{pmatrix} 1 & \\ & c \end{pmatrix} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1/c \end{pmatrix}$$
What happens to the multiplier, γ?

21. After a rotation, show that
$$x'^{\,2} + y'^{\,2} = (x, y) \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$
i.e., length is invariant under rotation. After a Lorentz transformation (timespace shear), show that
$$t'^{\,2} - x'^{\,2} = (t, x)\gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = t^2 - x^2$$
i.e., something else is invariant under timespace shear. Since this is the time as experienced by the moving observer (for hem, x'=0) it is called "proper time" and written τ. (It is also called "interval".)

22. Using $t^2 - x^2 = \tau^2$, calculate the time intervals experienced by two travellers (suppose all accelerations are instantaneous):
Traveller A goes from $(t, x) =$ (0,0) to (41,9) at constant speed 9/41 lights. Traveller B waits at home from $(t, x) =$ (0,0) to (23,0), then follows A at 5/13 lights to (36,5), then B accelerates to 4/5 lights to reach (41,9) coincidentally with A. (www.math.uic.edu/∼fields/puzzle/triples.html gives an alternative to calculating square roots.)
Draw their trajectories on a timespace diagram, marking the coordinates for each event and the proper times for each straight-line segment. Also mark the coordinates in seconds and

gigameters.
Is it generally true that the straight-line traveller experiences a greater time interval than the meandering traveller? (Ref.: [TW92, p.154, sects. 5.6, 5.7])

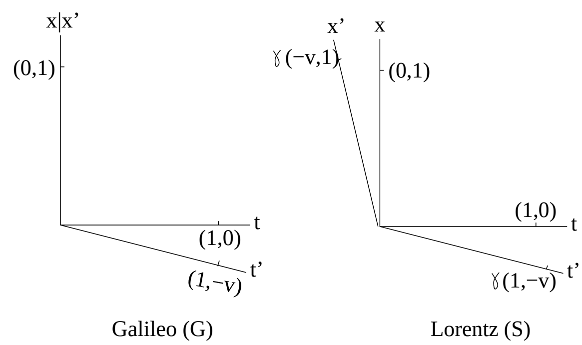
23. What is the "proper time", τ, between the "event" (36,5) in the previous question, and a new event, (21,22)? Discuss this value in terms of the convention for plotting two-dimensional numbers (see Week 4).

24. a) Plot (by hand or somehow by computer) the following points on a timespace diagram.

$$\begin{pmatrix} t \\ s \end{pmatrix} = \begin{array}{cccccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} \\ \begin{pmatrix} -5 \\ 4 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 4 \\ 5 \end{pmatrix} & \begin{pmatrix} 9 \\ 1 \end{pmatrix} & \begin{pmatrix} -4 \\ -4 \end{pmatrix} & \begin{pmatrix} 5 \\ -5 \end{pmatrix} & \begin{pmatrix} -1 \\ 10 \end{pmatrix} & \begin{pmatrix} 8 \\ 9 \end{pmatrix} \end{array}$$

b) On your plot, connect the dots as follows to show the "worldlines" of three particles: a-b-c-d, e-b-f and g-c-h. Describe what is happening, including any special or strange aspects.
c) Calculate the new positions of these points under the Lorentz transformation of an observer travelling at 0.9 lights, plot them and the transformed worldlines, and again describe all aspects of your result.
d) Confirm that proper times are invariant under the Lorentz transformation.

25. The lecture notes say that "both Galileo and Lorentz give new time axes to the traveller", but we know that the Galilean traveller experiences the same time as the Galilean non-traveller. How should we use the diagrams



to understand the meaning of this?

26. An example of two anti-Pythagorean quantities is the hyperbolic trigonometric pair of functions, $\cosh^2(u) - \sinh^2(u) = 1$ Look them up and write up their properties.

27. Another example of two anti-Pythagorean quantities is the ordinary trigonometric pair, $\sec^2(\theta) - \tan^2(\theta) = 1$ Prove this, and show that

$$\begin{pmatrix} x \\ y' \end{pmatrix} = \begin{pmatrix} \sec & -\tan \\ -\tan & \sec \end{pmatrix} \begin{pmatrix} x' \\ y \end{pmatrix}$$

if

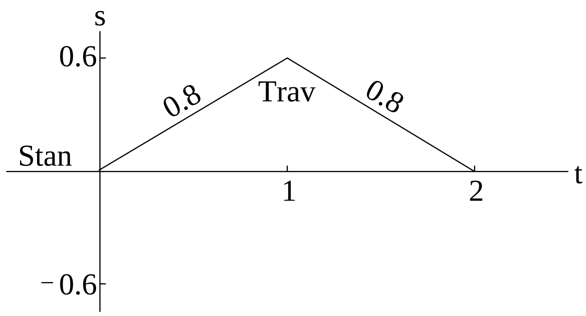
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

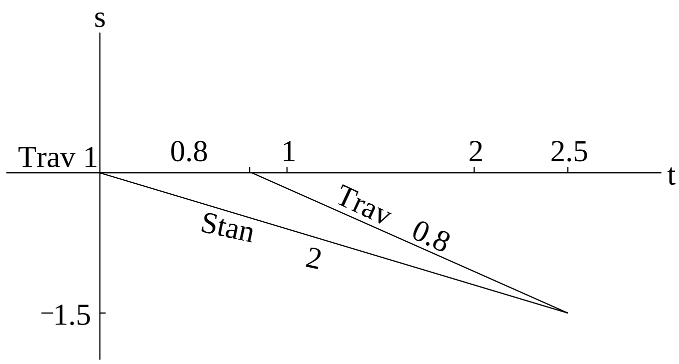
Note $x^2 + y^2 = x'^{\,2} + y'^{\,2}$ so $x^2 - y'^{\,2} = x'^{\,2} - y^2$.

28. **The "twin paradox".** If Trav heads off to his favorite comet in the Oort cloud at 0.6 lightspeed, then turns around and comes back at the same speed (but reversed), his clock and all processes aboard his spaceship will have slowed down and he will have aged less than his twin, Stan, whom he left behind on Earth.
Of course, relatively speaking, we could say that Stan and Earth moved at 0.6 lightspeed away from Trav in the opposite direction, leaving Trav in his spaceship standing still (while the Oort cloud rushes toward him at 0.6 lights).
Why then does Stan not age less than Trav?
The process is asymmetrical because Trav turns around and comes back.
We can do the calculation from both viewpoints, using antiPythagoras for the proper time τ

$$\tau = \sqrt{t^2 - s^2}$$

for an interval of spatial length s and temporal duration t.





The upper figure shows Stan's viewpoint: Trav moves away from him at 0.6 lights for 1 year, reaching the middle of the Oort cloud at 0.6 lightyears or 40,000 radii of Earth's orbit; then Trav returns, arriving after a total absence of two years. The time that Trav perceives—and that all the processes aboard his spaceship, mechanical or biological, measure—is for each leg

$$\sqrt{1^2 - 0.6^2} = 0.8$$

So while Stan aged two years, Trav aged only 1.6.
The lower figure shows the same events from Trav's point of view. He is sitting in his spacsehip going nowhere while the Earth, with Stan, rushes away from him and the Oort cloud rushes toward him. After 0.8 years (see above) the Oort cloud reaches him and he turns around to go home. Since Earth is moving away at 0.6 lights, Trav must really scamper to catch up. From Stan's point of view he must change from 0.6 lights outgoing to -0.6 lights homegoing. This is not a change of 1.2 lights but, as we saw in an earlier Excursion,

$$-\frac{u+v}{1+uv} = -\frac{1.2}{1+0.36} = -\frac{15}{17}$$

I've drawn the line showing this speed and where it intersects Stan's line at -0.6 lights, at $t = 2.5$ years and $s = -1.5$ lightyears according to Trav's outgoing viewpoint. But the time Trav sees on that leg is

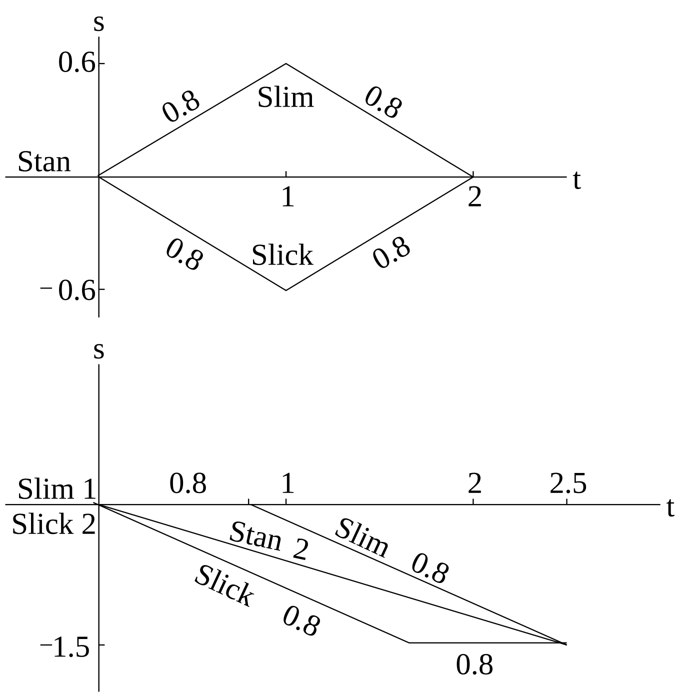
$$\sqrt{(2.5 - 0.8)^2 - 1.5^2} = 0.8$$

And the time Stan sees between leaving Trav and meeting him again is

$$\sqrt{2.5^2 - 1.5^2} = 2$$

These are exactly what we found from Stan's point of view, in the upper diagram.
What happens if we generalize to any velocity v? To different velocities, u (outgoing) and v (homegoing)?
We could investigate a third viewpoint, that of Trav homegoing. What happens?

29. **"Triplet paradox".** What if, instead of twins, Stan and Trav, we had triplets, Stan, Slim and Slick? Stan as before stays home but Slim and Slick have favourite comets at diametrically opposite locations in the Oort cloud. So they go off in opposite directions.
Because there is symmetry between Slim and Slick they age the same, but both age more slowly than Stan.



You can do the same calculations as in the previous Excursion, to get the results shown in the diagram.
Incidentally, this discussion points to the answer to the question posed at the end of the previous Excursion: the lower figure shows Slick's homegoing viewpoint. You can rearrange the triangle to show Slim's homegoing perspective.

30. The area of the parallelopiped formed by two vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix}$$

is the determinant of the matrix formed by these two vectors, say

$$\begin{vmatrix} a & d \\ b & e \end{vmatrix}$$

This is also the product of the lengths of the two vectors and the sine of the angle between them.
a) Confirm these statements.
b) If we think of this math as the "area operator", A, on the two vectors, u and v, is it commutative ($uAv - vAu = 0$)? Anti-commutative ($uAv + vAu = 0$)?

31. If the speed of light is invariant (i.e., not changed by the Lorentz transformation), why is it also the maximum possible speed?

32. Use MATLAB to generate the 25 coordinate pairs

$$(x, y) = (-2, -2), (-2, -1), .., (0, 0), .., (2, 2),$$

to shear them all through $v = 0.1$ with the Lorentz transformation, S

$$(u, w) = S(x, y)$$

and to display the shear as arrows from each (x, y) to the corresponding (u, w). Try it again with the Galilean shear transformation.

33. Look up Galileo Galilei, 1564–1642. What principle of relativity did he state?

34. Look up Albert Einstein, 1879–1955, and find out what questions he was wondering about (from age about 16 to 26) that led to special relativity. What did he come to think about his quantum theory later on as others developed it?

35. Look up Hendrik Antoon Lorentz, 1853–1928. What motivated him to invent the Lorentz contraction?

36. Show that the interval between an event, e, and the origin, (0,0), is pure time for Travis if and only if it appears to Stanley to lie on Travis' "worldline", $e = p(1, v)$, for some number p; and that it is pure space for Travis if and only if it appears to Stanley to lie on Travis' "co-worldline", $e = p(v, 1)$, again for some number p. Note that the arguments leading to space contraction and time dilation in Note 10 use, respectively, simultaneity (pure space) and "co-spaciality" (pure time), in Travis' frame. Show that the contraction and dilation factors are each $\sqrt{1 - v^2}$.
Give the corresponding arguments for Stanley's contraction and dilation from Travis' point of view (by reversing the sign on v).

37. View the video (you need a high-speed link and Flash Player 6)
www.onestick.com/relativity
and explain what you learned. Calculate the "time dilation" and the "Lorentz contraction" for a spaceship moving at $0.995c$, and the speed, relative to Earth, of the toy ship launched at $.995c$ from the spaceship going at $.995c$.

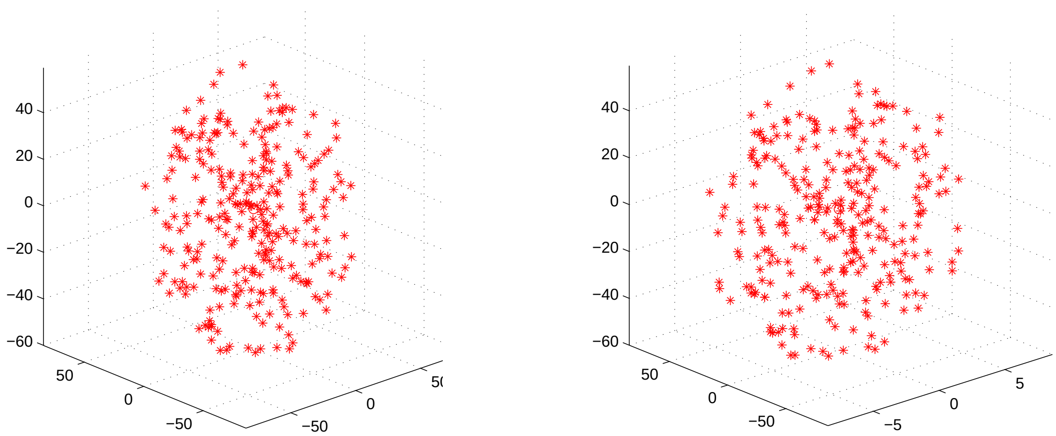
38. Using the constancy of lightspeed, Pythagoras and the triangle of velocities shown in
www.onestick.com/relativity,
show that the time of the traveller appears to the "stationary" observer to be slowed down by a factor $1/\gamma = \sqrt{1 - (v/c)^2}$.

39. Read the chapter “City Speed Limit” in [Gam39], where Gamow imagines lightspeed to be about 15 km/h. Show some of the events in that story on a timespace diagram and use the Lorentz transformation to calculate the amounts of space contraction and time dilation described.

40. On a timespace diagram, show the event, midnight of Dec. 31, 1999 in Ottawa, as perceived by observers in Vancouver, Calgary, Winnipeg, Montreal, Halifax and Newfoundland. (You do not need to know lightspeed for this.)

41. The horizontal lines depicting the “worldlines” of the planets in the timespace diagrams of both Notes 3 and 9 are shown in Stan’s frame. How would they look in Trav’s frame in each diagram?
Could Trav, parting in another direction at 17:00, get to Jupiter in time to see the comet strike? to Titan in time for Huygens’ landing?

42. Here are the stars closest to the Sun from the excursion of Week 1 (left), and the same star field as seen from the point of view of a spaceship almost at the Sun travelling along the x-direction at 99% of lightspeed (right). (The axes are measured in lightyears.)



What is the significant difference?
Adapt the MATLAB program from Week 1 for drawing the field of stars closest to the Sun to show them as they appear at an appreciable fraction of lightspeed.

43. Discuss Feynman’s demonstration [FLS64, Sect. 13-6] that magnetism is the result of electicity and relativity.

44. A newspaper story on Jan. 11, 2007 reported a prediction that in about 1000 years we will see a new supernova in the Eagle Nebula. Since the nebula is 7000 lightyears away, the scientists are saying that the star went supernova 6000 years ago. Discuss the possibility that theory can travel faster than light.

45. Any part of the Preliminary Notes that needs working through.

References

[FLS64] R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume II.* Addison-Wesley, 1964.

[Gam39] G. Gamow. *Mr Tompkins in Wonderland*. Cambridge University Press, London, 1939.

[TW92] Edwin F. Taylor and John Archibald Wheeler. *Spacetime Physics: Introduction to Special Relativity*. W. H. Freeman and Co., New York, 1992.