The Number of Lines Tangent to Arbitrary Convex Polyhedra in 3D

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ABSTRACT

We prove that the lines tangent to four possibly intersecting convex polyhedra in $\mathbb{R}^3$ with n edges in total form $\Theta(n^2)$ connected components in the worst case. In the generic case, each connected component is a single line, but our result still holds for arbitrary degenerate scenes. More generally, we show that a set of k convex polyhedra with a total of n edges admits, in the worst case, $\Theta(n^2k^2)$ connected components of (possibly occluded) lines tangent to any four of these polyhedra. We also show a lower bound of $\Omega(n^2k^2)$ on the number of non-occluded maximal line segments tangent to any four of these k convex polyhedra.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations; G.2.1 [**Combinatorics**]: Counting problems; I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling; I.3.7 [**Three-Dimensional Graphics and Realism**]: Visible line/surface algorithms

General Terms

Algorithms, Theory.

Keywords

Computational geometry, 3D visibility, visual events, visibility complex.

1. INTRODUCTION

Computing visibility relations in a 3D environment is a central problem of computer graphics (e.g. determining the view from a given point, identifying the set of blockers between two polygons, computing the umbra and penumbra

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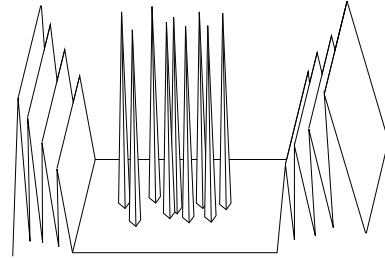[∗∗]Soongsil University, hsnaa@computing.ssu.ac.kr

Figure 1: A terrain of size $O(n)$ with $\Omega(n^4)$ maximal non-occluded line segments tangent in four points.

cast by an area light source) and other engineering tasks (e.g. radio propagation simulation). Visibility is well-known to account for a significant portion of the cost of the overall computation in these applications, and consequently a large body of research is devoted to speeding up visibility computations through the use of data structures (see [7] for a survey, including space partitions, aspect graphs, and visual hulls).

One such structure, the visibility complex [11], which has been proposed for rendering [9], encodes visibility relations by partitioning the set of maximal free line segments. Unfortunately the complex is potentially enormous which has so far prevented its application in practice. Its size is intimately related to the number of lines tangent to four objects in the scene; for n triangles in $\mathbb{R}^3$, the complex can have size $\Theta(n^4)$ in the worst case [6], even when the triangles form a terrain (see Figure 1).

There is evidence, both theoretical [5] and practical [8], that this bound is largely pessimistic in practical situations. The lower bound examples are carefully manufactured to exhibit the worst-case behavior, and often require an unrealistically large ratio between longest and shortest edges. On the other hand, for random scenes, Devillers et al. [5] prove that the expected size of the visibility complex is much smaller; for uniformly distributed unit balls the expected size is linear and for polygons or polyhedra of bounded aspect ratio it is at most quadratic.

While these results are encouraging, most scenes are not random. In fact, most scenes have a lot of structure which we can exploit; a scene is typically represented by many triangles which form a much smaller number of convex patches. If a scene consists of k *disjoint* convex polyhedra with n edges in total then, under a strong general position assumption, the number of tangents to four of the polyhedra is at most $O(n^2k^2)$; this follows directly from the bound proved in [10] on the number of combinatorial changes of the sil-

houette map viewed from a point moving along a straight line, and was also later proved in [3].

Our results. In this paper, we generalize the result of [3, 10] in two ways. First, we consider polytopes (i.e., convex polyhedra) that may **intersect**. We show that among k polytopes of total complexity n, the number of lines tangent to any four of them is in the worst case either infinite or $\Theta(n^2 k^2)$. The most surprising aspect of this result is that the bound (which is tight) is the same whether the polytopes intersect or not [3]. This is in sharp contrast to the 2D case, where the number of tangents of two convex polygons is always 4 if disjoint, and could be linear in the size of the polygons if they intersect.

Secondly we consider polytopes in **arbitrary position**: we drop all general position assumptions. Four such objects may admit an infinite number of common tangents which can be partitioned in line space into connected components. Two lines are said to be in the same *connected component* if and only if their corresponding points in line space (e.g., Plücker space) are in the same connected component of the set of points corresponding to all the lines tangent to any 4 among the k objects; equivalently one line can be continuously moved into the other while remaining tangent to at least four objects[1].

Our main results are the following.

THEOREM 1. *Given 4 polytopes in $\mathbb{R}^3$ with n edges in total, the number of lines tangent to the 4 polytopes is, in the worst case, either infinite or $\Theta(n^2)$. More generally, these lines form $\Theta(n^2)$ connected components. Moreover, if one of the polytopes has m edges, these bounds improve to $\Theta(mn)$.*

This result improves the previously known upper bound of $O(n^3 \log n)$ which follows from the same bound on the complexity of the set of line transversals to a set of polyhedra (here four) of total complexity $O(n)$ [1].

THEOREM 2. *Given k polytopes in $\mathbb{R}^3$ with n edges in total, the lines (possibly occluded) tangent to any four of the polytopes form, in the worst case, $\Theta(n^2 k^2)$ connected components.*

This result improves the trivial bound of $O(n^4)$.

Our results hold for any set of polytopes including those that degenerate to a polygon, segment or point[2]. The polytopes may intersect in any way; they may overlap or coincide and their faces may be triangulated.

To emphasize the importance of considering intersecting polytopes, observe that computer graphics scenes usually consist of not necessarily convex objects which can, however, be decomposed into sets of convex polyhedra. Notice that simply decomposing these objects into convex polyhedra with disjoint interiors may induce a scene of much higher complexity than a decomposition into intersecting polytopes. Moreover, the decomposition of a polyhedron into interior-disjoint polytopes may yield new tangents which

Figure 2: Plane Π_t contains edge e and intersects polytopes P, Q, and R in polygons P_t, Q_t, and R_t, respectively.

were not present in the original scene; indeed a line tangent to two polytopes along a shared face is not tangent to their union.

The importance of considering polytopes in arbitrary position comes from the fact that graphics scenes are full of degeneracies both in the sense that four polytopes may admit infinitely many tangents and that polytopes may share edges or faces. Note that we could not find a perturbation argument that guarantees the preservation of all connected components of tangents and we do not believe it is a simple matter.

The paper is organized as follows. We prove the upper bounds of Theorems 1 and 2 in Sections 2 and 3, and the lower bounds in Section 4.

2. MAIN LEMMA

We prove in this section the following lemma which yields the upper bounds in Theorem 1. This lemma is also fundamental for the proof of the upper bound of Theorem 2, which we provide in Section 3.

Consider three polytopes **P**, **Q**, and **R** in $\mathbb{R}^3$, with p, q, and r edges, respectively, and let e be an edge of a fourth polytope **S**.

MAIN LEMMA. *The lines intersecting e and tangent to **P**, **Q**, **R** and **S** form $O(p + q + r)$ connected components.*

The upper bounds of Theorem 1 follow from the Main Lemma. Indeed, if s denotes the number of edges of **S**, by summing over all these edges, the number of connected components of lines tangent to the four polytopes is $O(s(p+q+r)) = O(n^2)$. Moreover, if one of the polytopes has m edges, choosing **S** to be this polytope yields that the lines tangent to the four polytopes form $O(mn)$ connected components.

The actual proof of the Main Lemma is rather complicated because it handles polytopes which may intersect in any way. In addition, the proof handles all the degenerate cases. To assist the reader, we first give an overview of the proof.

2.1 Proof overview

The proof is inspired by a method which was, to our knowledge, first used in [2] (see also Schiffenbauer's survey [12]).

We sweep the space with a plane Π_t rotating about the line containing e. The sweep plane intersects the three polytopes **P**, **Q**, and **R** in three, possibly degenerate or empty, convex polygons denoted P_t, Q_t, and R_t, respectively (see Figure 2).

During the sweep, we track the *bitangents*, that are the lines tangent to P_t and Q_t, or to Q_t and R_t. As the sweep

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[1]Notice that the set of objects the line is tangent to might change during the motion.

[2]A line is tangent to a polytope with non-empty interior if and only if it intersects the polytope but not its interior. We also use this definition if the polytope is a segment or a point; if the polytope is a polygon then a line is a tangent line if and only if it intersects the 2D boundary of the polygon.
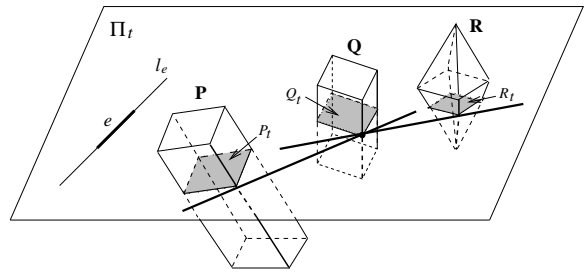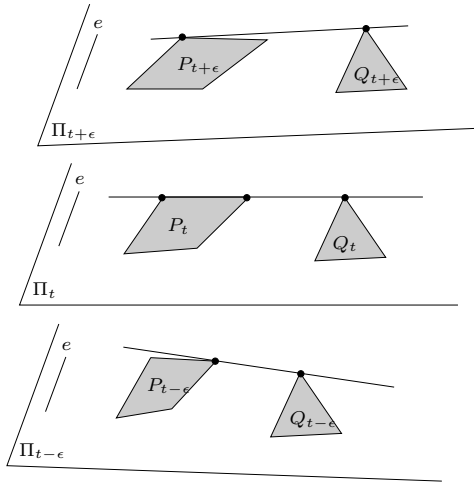
Figure 3: A bitangent to P_t and Q_t is tangent to P_t along an edge. The plane Π_t is F-critical.

plane rotates, the three polygons deform and the bitangents move accordingly. Every time two bitangents become aligned during the sweep, their supporting line is tangent to **P**, **Q**, and **R**. A bitangent to two polygons, say P_t and Q_t, goes through a set of vertices of these polygons. The edges of **P** and **Q** that contain these vertices are called *support edges* of the bitangent. (For the purpose of this overview we ignore the case where bitangents go through vertices of **P** or **Q**).

In any given instance of the sweep plane Π_t, we consider the pairs of bitangents (one involving P_t and Q_t, and the other Q_t and R_t) that share a vertex of Q_t (see Figure 2). The sets of edges that consist of e and the support edges of two such bitangents are called *candidate tuples*. A candidate tuple contains at least 4 edges, e and at least one edge from each polytope; most of the time, it contains exactly 4 edges.

A candidate tuple induced by an instance of the sweep plane changes as the plane rotates only when a support edge of a bitangent changes. We define *critical planes* in such a way that the support edges of the bitangents do not change as the sweep plane rotates in between two consecutive critical planes. As the sweep plane rotates, the support edges of a bitangent change if a support edge starts or ceases to be swept, or if, during its motion, the bitangent becomes tangent to one of the polygons along an edge (see Figure 3). We thus define two types of critical planes. An instance of the sweep plane is called *V-critical* if it contains a vertex of one of the polytopes, and *F-critical* if it contains a line that lies in the plane containing a face of one of the polytopes and is tangent to another of the polytopes (see Figures 3 and 4). We will show that the number of critical planes is $O(p+q+r)$.

Notice that when the polytopes intersect there may exist a linear number of bitangents in an instance of the sweep plane; two intersecting convex polygons may admit a linear number of bitangents. Thus there can be a linear number of candidate tuples induced by any instance of the sweep plane and the linear number of critical planes leads to a quadratic bound on the total number of distinct candidate tuples. In the detailed proof of the lemma (Section 2.2), we amortize the count of candidate tuples over all the critical planes to get a linear bound on the number of distinct candidate tuples

and thus on the number of connected components of lines intersecting e and tangent to **P**, **Q**, and **R**.

2.2 Proof of the Main Lemma

We now give a detailed proof of the Main Lemma for the general case of possibly intersecting polytopes in any configuration.

Since any line tangent to a polytope will remain tangent whether or not the faces are triangulated (or otherwise subdivided), we assume without loss of generality that *no two faces of* **P** *are coplanar, and similarly for* **Q**, **R** *and* **S**.

We can also assume without loss of generality that **P**, **Q**, **R** and **S** have *non-empty interior*. Indeed, if some polytopes have empty interior, they can be "inflated" as follows so that there are not more connected components of lines tangent to the four polytopes before the inflation than after[3]. If, say, **P** is a segment (resp. a point) it can be first transformed into any triangle having this segment (resp. point) as an edge (resp. vertex) without losing any common tangent line to the four polytopes. Now, if **P** is a polygon it can be "inflated" by adding a vertex that projects orthogonally onto **P**. Let **P′** and **P″** be two inflated instances of **P** for which the added vertices are added on opposite sides of **P** respectively. Then any line tangent to **P** is tangent to **P′** or **P″**.

Let l_e be the line containing e and let Π_t denote the sweep plane parameterized by $t \in [0, \pi]$ such that Π_t contains the line l_e for all t and $\Pi_0 = \Pi_\pi$. Each plane Π_t intersects the three polytopes **P**, **Q**, and **R** in three, possibly degenerate or empty, convex polygons, P_t, Q_t, and R_t, respectively (see Figure 2).

A line intersecting e and tangent to **P**, **Q**, **R** and **S** is called a *generic tangent line* if it is tangent to P_t, Q_t, and R_t in some plane Π_t. Otherwise it is called a *nongeneric tangent line*. A nongeneric tangent line properly[4] intersects P_t, Q_t, or R_t in some plane Π_t. In this case P_t, Q_t, or R_t is a face or an edge of **P**, **Q**, or **R** lying in Π_t, a degenerate situation.

We bound in the two following subsections the number of generic and non-generic tangent lines.

Generic tangent lines

We start with some definitions. We call *support vertex* of a line, a vertex of **P**, **Q**, or **R** that lies on the line. We call *support edge* of a line, an edge of **P**, **Q**, or **R** that intersects the line and has no endpoint on it. A *support* is a support edge or support vertex.

For any t, a *bitangent* to polygons P_t and Q_t is a line tangent to P_t and Q_t in Π_t (the line may intersect the polygon R_t in any way). For any t, let a (P_t, Q_t)-*tuple* be the nonordered set of all supports in **P** and **Q** of one of the bitangents to polygons P_t and Q_t. Note that a (P_t, Q_t)-tuple consists of exactly one support in **P** and one support in **Q** except when the corresponding bitangent is tangent to **P** (or **Q**) along a face; then the (P_t, Q_t)-tuple contains two supports in **P** (or **Q**) instead of one. A **PQ**-*tuple* is a set of edges and vertices that is a (P_t, Q_t)-tuple for some t. We define similarly the (Q_t, R_t)-*tuples* and **QR**-*tuples*.

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[3]This simple argument actually only holds for extremal tangent lines ; for the general argument, see the full paper.

[4]A line *properly* intersects a polygon if and only if it intersects its relative interior.
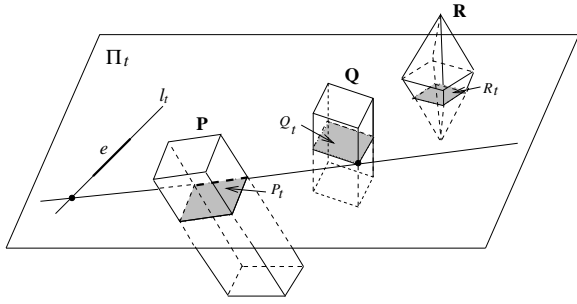
Figure 4: Plane Π_t contains a line that is tangent to Q and lies in a plane containing a face of P. Plane Π_t is F-critical.

We say that a (P_t, Q_t)-tuple is *maximal for some t* if it is not contained in any other (P_t, Q_t)-tuple, for the same t. Note that a (P_t, Q_t)-tuple is non-maximal for some t only if all its supports intersects Π_t in one and the same point (see Figure 6(b)).

For any t, let a (P_t, Q_t, R_t)-*tuple* be the union of a (P_t, Q_t)-tuple and a (Q_t, R_t)-tuple that share at least a support in **Q**. A (P_t, Q_t, R_t)-tuple is maximal for some t if it is not contained in any other (P_t, Q_t, R_t)-tuple, for the same t. A **PQR**-*tuple* is a set of edges and vertices that is a (P_t, Q_t, R_t)-tuple for some t. Note that a **PQR**-tuple typically consists of three supports, one from each polytope, and consists, in all cases, of at most two supports in **P**, at most three supports in **Q**, and at most two supports in **R**.

LEMMA 3. *The set of supports of a generic tangent line is a **PQR**-tuple.*

PROOF. Any generic tangent line ℓ is tangent in Π_t to P_t, Q_t, and R_t for some value t. Thus the set of supports of ℓ in **P** and **Q** (resp. in **Q** and **R**) is a (P_t, Q_t)-tuple (resp. a (Q_t, R_t)-tuple). Moreover these (P_t, Q_t)-tuple and (Q_t, R_t)-tuple contain the same supports in **Q**, thus their union is a (P_t, Q_t, R_t)-tuple, hence a **PQR**-tuple. $\square$

We now define the *critical planes* Π_t in such a way that the set of (P_t, Q_t, R_t)-tuples is invariant for t ranging strictly in between two consecutive critical values. We introduce two types of critical planes: the *V-critical* and *F-critical planes*.

A plane Π_t is *V-critical* if it contains a vertex of **P**, **Q**, or **R**, not[5] on l_e. A plane Π_t is *F-critical* relative to polytopes $(\mathbf{P}, \mathbf{Q})$ if (see Figure 4) it contains a line ℓ such that

(i) ℓ lies[6] in a plane $\Psi \neq \Pi_t$ containing a face of **P**, and

(ii) ℓ is tangent in Ψ to polygon[7] $\mathbf{Q} \cap \Psi$ or $\mathbf{P} \cap \Psi$, at some point not on l_e.

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[5] This constraint ensures that the number of V-critical planes is finite even in degenerate configurations.

[6] For simplicity, we do not require that ℓ is tangent to **P**; this leads to overestimating the number of common tangents to **P**, **Q**, **R**, and **S** but only by an asymptotically negligible amount.

[7] Note that not all lines in Ψ tangent to **Q** are tangent to the polygon $\mathbf{Q} \cap \Psi$ when that polygon is a face or edge of **Q** lying in Ψ. Note also that we define Π_t to be F-critical when ℓ is tangent to $\mathbf{P} \cap \Psi$ at some point not on l_e only for handling the very degenerate case where $\mathbf{Q} \cap \Psi$ is an edge of **Q**, and there exists a line in Ψ that properly intersects $\mathbf{Q} \cap \Psi$ and is tangent to $\mathbf{P} \cap \Psi$ along an edge that has an endpoint on l_e (see Figure 10 at the end of the paper).

F-critical planes relative to $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$, and $(\mathbf{R}, \mathbf{Q})$ are defined similarly. A plane Π_t is *F-critical* if it is F-critical relative to polytopes $(\mathbf{P}, \mathbf{Q})$, $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$, or $(\mathbf{R}, \mathbf{Q})$.

The values of t corresponding to critical planes Π_t are called *critical values*. We call *V-critical* and *F-critical events* the couples (t, o) where t is a critical value and o is a vertex or line depending on the type of critical event. In a V-critical event, o is a vertex of **P**, **Q**, or **R** that belongs to $\Pi_t \setminus l_e$. In an F-critical event, o is a line lying in some plane Π_t and satisfying Conditions (i-ii) above. A *critical event* is a V-critical or F-critical event.

LEMMA 4. *There are at most $\frac{2}{3}(p+q+r)$ V-critical events and $\frac{8}{3}(p + 2q + r)$ F-critical events.*

PROOF. The number of V-critical events is at most the total number of vertices of **P**, **Q**, and **R**, and hence is less than two thirds the total number of edges of **P**, **Q**, and **R**.

We now count the number of F-critical events relative to polytopes $(\mathbf{P}, \mathbf{Q})$. Let Ψ be a plane containing a face of **P**. Let ℓ be a line lying in some plane Π_t and satisfying with plane Ψ Conditions (i-ii). Plane Ψ does not contain l_e because otherwise both l_e and ℓ lie in the two distinct planes Ψ and Π_t, so $\ell = l_e$ but then ℓ cannot satisfy Condition (ii). Furthermore ℓ and l_e intersect or are parallel since they both lie in Π_t. Thus if $\Psi \cap l_e$ is a point then ℓ contains it, and otherwise $\Psi \cap l_e = \emptyset$ and ℓ is parallel to l_e.

There are at most four lines ℓ in plane Ψ going through $\Psi \cap l_e$ if this intersection is a point and parallel to l_e otherwise, and tangent to $\mathbf{Q} \cap \Psi$ or $\mathbf{P} \cap \Psi$, at some point not on l_e. Moreover each such line is contained in a unique plane Π_t, for $t \in [0, \pi]$, since $\ell \neq l_e$ (ℓ contains a point not on l_e). Hence, a face of **P** generates at most four F-critical events relative to $(\mathbf{P}, \mathbf{Q})$. Therefore the number of critical events relative to $(\mathbf{P}, \mathbf{Q})$ is at most $\frac{8}{3}p$ since the number of faces of a polytope is at most two thirds the number of its edges. Hence the number of critical events relative to $(\mathbf{P}, \mathbf{Q})$, $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$ and $(\mathbf{R}, \mathbf{Q})$ is at most $\frac{8}{3}(p + 2q + r)$. $\square$

We now prove that the critical planes have the desired property. Let u_e be the set of supports of l_e in **P** and **Q**.

LEMMA 5. *If $u \neq u_e$ is a maximal[8] (P_t, Q_t)-tuple for some but not all t in every open neighborhood of t^*, then t^* is a critical value.*

Moreover, there exists a V-critical event (t^*, v) or a F-critical event (t^*, m) such that u contains v or an edge with endpoint v, or u is contained in the set of supports of m.

PROOF. We can assume that there exists an open neighborhood $\mathcal{N} = (t_0, t_1)$ of t^* such that u is a maximal (P_t, Q_t)-tuple for $t = t^*$ or all $t \in (t^*, t_1)$ (or by symmetry for all $t \in (t_0, t^*)$), but is not a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}$. Indeed, suppose that Lemma 5 is proved if u satisfies this additional assumption. If u does not satisfy this additional assumption, then there exists, in every open neighborhood of t^*, infinitely many maximal intervals (open or closed and possibly of zero length) in which either u is a maximal (P_t, Q_t)-tuple for all t or u is not a maximal (P_t, Q_t)-tuple for all t. However, in this case, our assumption that Lemma 5 holds if u satisfies the above additional

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[8] For simplicity, we prove Lemma 5 under the assumptions that u is maximal and distinct from u_e; note however that the lemma remains correct without these assumptions.

assumption yields that each endpoint of these intervals is a critical value, contradicting Lemma 4.

We can also assume that u *contains no vertex* v, *or edge with endpoint* v, *such that* v *lies on* $\Pi_{t^*} \setminus l_e$ because otherwise (t^*, v) is a V-critical event such that u contains v or an edge with endpoint v, which concludes the proof.

We prove a series of claims that yields the result. Indeed, we prove the existence of a line m in Π_{t^*} whose set of supports contains u (Claim C) such that (i) m lies in a plane $\Psi \neq \Pi_{t^*}$ containing a face of $\mathbf{P}$ (Claim D), and (ii) m is tangent in Ψ to polygon $\mathbf{Q} \cap \Psi$ or $\mathbf{P} \cap \Psi$, at some point not on l_e (Claim E). This proves that Π_{t^*} contains a line m whose set of supports contains u and such that (t^*, m) is an F-critical event, which concludes the proof.

We only consider in the following supports in $\mathbf{P}$ and in $\mathbf{Q}$; polytope $\mathbf{R}$ plays no role. We start by proving two preliminary facts.

CLAIM A. *Each support in* u *intersects* Π_t *in exactly one point (possibly on* l_e*), for all* t *in any sufficiently small open neighborhood* $\mathcal{N}$ *of* t^**.*

Moreover, the union of all supports in u *intersects* Π_t *in at least two distinct points for all* $t \neq t^*$ *in* $\mathcal{N}$*. This property also holds for* $t = t^*$ *if* u *is a maximal* (P_{t^*}, Q_{t^*})*-tuple.*

PROOF OF CLAIM A. Since u is a (P_t, Q_t)-tuple for some t in every open neighborhood of t^*, each support in u intersects Π_t for some t in every open neighborhood of t^*. It thus follows from the assumption that u contains no vertex v, or edge with endpoint v, such that v lies on $\Pi_{t^*} \setminus l_e$ that each support in u intersects Π_t for all t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*. It follows that each support in u either lies in l_e or intersects Π_t in exactly one point for all $t \in \mathcal{N}$. However, no edge of u lies in l_e because otherwise, if x denotes such an edge of, say, $\mathbf{P}$, then any line tangent to P_t in Π_t and intersecting x contains an endpoint of x which is a vertex of $\mathbf{P}$; thus, by definition, u does not contain x but one of its endpoints. Hence each support of u intersects Π_t in exactly one point for all $t \in \mathcal{N}$.

We now prove that the union of the supports in u intersects Π_t in at least two distinct points for any $t \in \mathcal{N}$ such that u is a maximal (P_t, Q_t)-tuple. Suppose for a contradiction that the union of the supports in u intersects Π_t in one single point v for some $t \in \mathcal{N}$ such that u is a maximal (P_t, Q_t)-tuple. Then polygons P_t and Q_t are both reduced to point v because otherwise u is not maximal (indeed a line in Π_t tangent to P_t and Q_t at v can be rotated about v until it becomes tangent to P_t or Q_t in some other points). Thus v is a vertex of $\mathbf{P}$ and of $\mathbf{Q}$ because the polytopes have non-empty interior. It follows that v is a support in u. Thus vertex v lies on l_e since each support in u intersects Π_t for all $t \in \mathcal{N}$. Moreover, since P_t and Q_t are both reduced to point v, the set of supports of l_e is u, contradicting the hypotheses of Lemma 5.

Thus, if u is a maximal (P_t, Q_t)-tuple for all $t \in (t^*, t_1)$, the union of the supports in u intersects Π_t in at least two distinct points for all $t \in (t^*, t_1)$ and thus for all $t \neq t^*$ in any sufficiently small open neighborhood of t^*. Also, if u is a maximal (P_t, Q_t)-tuple for $t = t^*$, the union of the supports in u intersects Π_t in at least two distinct points for $t = t^*$ and thus for all t in any sufficiently small open neighborhood of t^*. □

CLAIM B. *If* u *is a maximal* (P_{t^*}, Q_{t^*})*-tuple then* u *consists of at least three supports.*
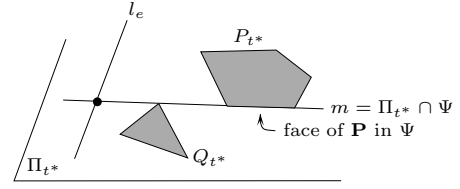

Figure 5: m **is tangent to** $\mathbf{P}$ **along a face in plane** $\Psi \neq \Pi_{t^*}$**.**

PROOF OF CLAIM B. Suppose for a contradiction that u consists of only two supports. It follows from Claim A that they intersect Π_t in exactly two distinct points for all t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*. Thus there exists for all $t \in \mathcal{N}$ a unique line m_t in Π_t whose set of supports contains u; moreover m_t is continuous in terms of t. Since u is a (P_{t^*}, Q_{t^*})-tuple, the set of supports of m_{t^*} is u. Thus, for all t in any sufficiently small $\mathcal{N}$, the set of supports of m_t is u. It follows, since m_{t^*} is tangent to P_{t^*} and Q_{t^*}, that m_t is tangent to P_t and Q_t for all $t \in \mathcal{N}$.

Hence, for all $t \in \mathcal{N}$, line m_t, whose set of supports is u, is tangent to P_t and Q_t in Π_t. Thus u is a (P_t, Q_t)-tuple for all $t \in \mathcal{N}$. Moreover, m_t is the unique line in Π_t whose set of supports contains u, thus u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}$, contradicting the hypotheses of the lemma. □

CLAIM C. *There exists a line* m *in* Π_{t^*} *whose set of supports contains* u *that is tangent to* P_{t^*} *and* Q_{t^*} *along an edge of one of them, say of* P_{t^*}*.*

PROOF OF CLAIM C. Consider first the case where u is a maximal (P_{t^*}, Q_{t^*})-tuple. There exists in Π_{t^*} a line m tangent to P_{t^*} and Q_{t^*} whose set of supports is u. By Claim B, the set u of supports of m contains at least two supports in $\mathbf{P}$ (or in $\mathbf{Q}$). Furthermore, the supports of m in one polytope intersect Π_{t^*} in distinct points (by definition of supports). Thus m intersects P_{t^*} (or Q_{t^*}) in at least two distinct points and is tangent to P_{t^*} and Q_{t^*}. The result follows since P_{t^*} (and Q_{t^*}) is convex.

Consider now the case where u is a maximal (P_t, Q_t)-tuple for all $t \in (t^*, t_1)$. Then, for all $t \in (t^*, t_1)$, there exists a line in Π_t tangent to P_t and Q_t and whose set of supports is u. Moreover, by Claim A, this line is unique for each $t \in (t^*, t_1)$ and varies continuously in terms of $t \in (t^*, t_1)$. When t tends to t^*, the line tends to a line m in Π_{t^*} which is tangent to P_{t^*} and Q_{t^*} and whose set of supports contains u. If its set of supports strictly contains u then m is tangent to P_{t^*} and Q_{t^*} along an edge of one of them because the polygons are convex. Otherwise, u is a (P_{t^*}, Q_{t^*})-tuple.

We can suppose that u is a non-maximal (P_{t^*}, Q_{t^*})-tuple since we already treated the case where u is maximal. There exists in Π_{t^*} a line tangent to P_{t^*} and Q_{t^*} whose set of supports is u. Since u is non-maximal this line is tangent to P_{t^*} and Q_{t^*} at a shared vertex, and can be rotated about this vertex in Π_{t^*} until it gets tangent to P_{t^*} and Q_{t^*} in some other points; let m denote the resulting line. The set of supports of m contains u and m is tangent to P_{t^*} and Q_{t^*} along an edge of one of them because the polygons are convex. □

CLAIM D. *Line* m *lies in a plane* $\Psi \neq \Pi_{t^*}$ *containing a face of* $\mathbf{P}$*.*

PROOF OF CLAIM D. By Claim C, m contains an edge of P_{t^*}; see Figure 5. This edge either intersects the relative
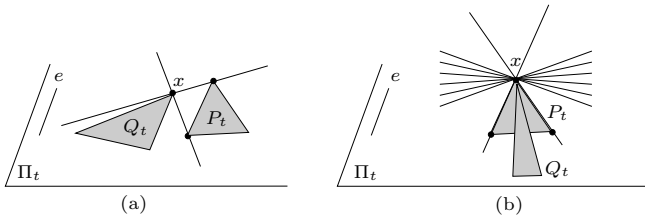
Figure 6: Lines through x in Π_t and tangent to P_t and Q_t.

interior of some face of $\mathbf{P}$ in which case we take Ψ to be the plane containing that face, or it is an edge of $\mathbf{P}$ in which case we take Ψ to be a plane, different from Π_{t^*}, containing one of the two faces of $\mathbf{P}$ incident to that edge. □

CLAIM E. *Line m is tangent to $\mathbf{P} \cap \Psi$ or to $\mathbf{Q} \cap \Psi$, at some point not on l_e.*

PROOF OF CLAIM E. By Claim C, m is tangent to Q_{t^*} and thus to $\mathbf{Q}$. It follows that m is tangent to $\mathbf{Q} \cap \Psi$ or properly intersects it, in which case $\mathbf{Q} \cap \Psi$ is a face or an edge of $\mathbf{Q}$. If m is not tangent to $\mathbf{Q} \cap \Psi$ at some point not on l_e then m is tangent to $\mathbf{Q} \cap \Psi$ only on l_e, or $\mathbf{Q} \cap \Psi$ is a face or an edge of $\mathbf{Q}$. These two situations are not generic since two edges of two distinct polytopes are then coplanar (in the first case an edge of $\mathbf{Q}$ and e are coplanar, and in the later case a face of $\mathbf{P}$ is coplanar with a face or an edge of $\mathbf{Q}$). Hence proving this claim is straightforward under some general position assumption, however for polytopes in arbitrary position the proof is rather intricate and thus postponed to the appendix for the sake of clarity.□

This concludes the proof of Lemma 5. ■

Let $\mathcal{I}$ be any interval of $\mathbb{R}/\pi\mathbb{Z}$ (open or closed, reduced to a point or not).

LEMMA 6. *Any edge or vertex of $\mathbf{P}$ or $\mathbf{Q}$ is in at most 2 $\mathbf{PQ}$-tuples that are maximal (P_t, Q_t)-tuples for all t in $\mathcal{I}$.*

PROOF. Let $\tilde{t} \in \mathcal{I}$ and x be an edge or vertex of $\mathbf{P}$ or $\mathbf{Q}$. If x does not intersect $\Pi_{\tilde{t}}$ then no $(P_{\tilde{t}}, Q_{\tilde{t}})$-tuple contains x. If x intersects $\Pi_{\tilde{t}}$ in one point then there is, in general, at most two lines in $\Pi_{\tilde{t}}$ going through x and tangent to $P_{\tilde{t}}$ and $Q_{\tilde{t}}$ (see Figure 6(a)); in all cases there are at most 3 $(P_{\tilde{t}}, Q_{\tilde{t}})$-tuples containing x (see Figure 6(b)), however at most 2 of them are maximal. If x intersects $\Pi_{\tilde{t}}$ in more than one point, x is an edge lying in $\Pi_{\tilde{t}}$. Then any line in $\Pi_{\tilde{t}}$ intersecting x and tangent to $P_{\tilde{t}}$ and $Q_{\tilde{t}}$ contains an endpoint of x and thus x belongs to no $(P_{\tilde{t}}, Q_{\tilde{t}})$-tuple.

Hence at most 2 $\mathbf{PQ}$-tuples contain x and are maximal (P_t, Q_t)-tuples for $t = \tilde{t}$ and thus for all t in $\mathcal{I}$. □

LEMMA 7. *There are at most $O(p + q + r)$ $\mathbf{PQR}$-tuples.*

PROOF. In order to count the number of distinct (P_t, Q_t, R_t)-tuples, we charge each maximal (P_t, Q_t, R_t)-tuple to a critical event. We then show that each critical event is charged at most a constant number of times. It then follows from Lemma 4 that there are $O(p + q + r)$ distinct maximal (P_t, Q_t, R_t)-tuples. A maximal (P_t, Q_t, R_t)-tuple consists of at most two supports in $\mathbf{P}$, at most three supports in $\mathbf{Q}$, and at most two supports in $\mathbf{R}$, and thus contains at most $3 \times 7 \times 3$ distinct subsets with at least one support in each of

$\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. Each maximal (P_t, Q_t, R_t)-tuple thus contains at most $3 \times 7 \times 3$ distinct (P_t, Q_t, R_t)-tuples which implies the result.

Let s be a maximal (P_t, Q_t, R_t)-tuple and let $\mathcal{I}$ be a maximal connected subset of $\mathbb{R}/\pi\mathbb{Z}$ such that s is a maximal (P_t, Q_t, R_t)-tuple for all $t \in \mathcal{I}$. Let u be a maximal (P_t, Q_t)-tuple and u' a maximal (Q_t, R_t)-tuple whose union is s and who share at least a support in $\mathbf{Q}$.

If $\mathcal{I} = \mathbb{R}/\pi\mathbb{Z}$ then u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{I}$. Since u is maximal, the union of the supports in u intersects Π_t in at least two distinct points. Moreover, each support in u intersects Π_t for all $t \in \mathcal{I}$ and thus intersects l_e. It follows that l_e is the only line in Π_t whose set of supports is u, for any $t \in \mathcal{I}$. Similarly for u'. Thus s is the set of supports of l_e. We can thus assume in the following that $\mathcal{I} \neq \mathbb{R}/\pi\mathbb{Z}$, and only count the maximal (P_t, Q_t, R_t)-tuples that are not the set of support of l_e.

Interval $\mathcal{I}$ is thus a non-empty interval of $\mathbb{R}/\pi\mathbb{Z}$; it can be open or closed, reduced to a point or not. Let w_0 and w_1 denote its endpoints.

If s contains a vertex v, or an edge with endpoint v, such that v lies in $\Pi_{w_i} \setminus l_e$, for $i = 0$ or 1, then we charge s to the V-critical event (w_i, v). Otherwise, we charge s to a F-critical event (w_i, m) where m is a line in Π_{w_i} whose set of supports contains u or u'. Such a V-critical or F-critical event exists by Lemma 5.

We now prove that each critical event is charged by at most a constant number of distinct maximal (P_t, Q_t, R_t)-tuples. As mentioned before, that will imply the result.

Consider a V-critical event (t^*, v) that is charged by a maximal (P_t, Q_t, R_t)-tuple s. By the charging scheme, s contains a support x that is v or an edge with endpoint v, and s is a maximal (P_t, Q_t, R_t)-tuple for all t in at least one of three intervals, $\{t^*\}$ and two open intervals having t^* as endpoint; denote by $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ these intervals.

By Lemma 6, at most 2 $\mathbf{PQ}$-tuples contain x and are maximal (P_t, Q_t)-tuples for all t in $\mathcal{I}_i$. Moreover, each of these $\mathbf{PQ}$-tuples contains at most 2 supports in $\mathbf{Q}$, and each of these supports belongs to at most 2 $\mathbf{QR}$-tuples that are maximal (Q_t, R_t)-tuples for all t in $\mathcal{I}_i$. Thus at most 8 $\mathbf{PQR}$-tuples contain x and are maximal (P_t, Q_t, R_t)-tuples for all t in $\mathcal{I}_i$, for each $i = 1, \ldots, 3$. Hence any V-critical event (t^*, v) is charged by at most 24 distinct maximal (P_t, Q_t, R_t)-tuples.

Consider now a F-critical event (t^*, m) that is charged by a maximal (P_t, Q_t, R_t)-tuple s, and define as before u and u'. By the charging scheme, the set of supports of m contains u or u' (or both); suppose without loss of generality that it contains u. The set of supports of m contains at most two supports in $\mathbf{P}$ and at most two supports in $\mathbf{Q}$. Since u contains at least one support in $\mathbf{P}$ and at least one support in $\mathbf{Q}$, there are at most 3^2 choices for u.

By the charging scheme, s is a maximal (P_t, Q_t, R_t)-tuple for all t in at least one of 3 intervals, $\{t^*\}$ and two open intervals having t^* as endpoint; denote by $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ these intervals. It follows from Lemma 6 that, for each support x of $\mathbf{Q}$ in u, at most 2 $\mathbf{QR}$-tuples contain x and are maximal (Q_t, R_t)-tuples for all t in $\mathcal{I}_i$. There are at most 3^2 choices for u, 2 for x, 3 for i and 2 for the $\mathbf{QR}$-tuples containing x. Hence any F-critical event (t^*, m) is charged by at most $2^2 \times 3^3$ distinct maximal (P_t, Q_t, R_t)-tuples.

Therefore each critical event is charged by at most a constant number of distinct maximal (P_t, Q_t, R_t)-tuples, which concludes the proof. □

We now show in the next two lemmas that there are at most $O(p+q+r)$ connected components of generic tangent lines.

LEMMA 8. *The set of transversals to e and to the edges and vertices of a **PQR**-tuple contains at most $O(1)$ connected components of tangents to **P**, **Q**, **R** and **S**.*

PROOF. The set of transversals to μ segments (possibly of zero length) consists of at most μ connected components [4]. Since the union of $\{e\}$ and any **PQR**-tuple consists of at most 8 segments (possibly of zero length), the result follows if the set of transversals to e and to the edges and vertices of a **PQR**-tuple is finite. We omit here the proof in the case where this set of transversals is not finite. $\square$

PROPOSITION 9. *The generic tangent lines form $O(p+q+r)$ connected components.*

PROOF. A generic tangent line is transversal to e and to the edges and vertices of a **PQR**-tuple, by definition and Lemma 3. Moreover, there are $O(p+q+r)$ **PQR**-tuples by Lemma 7. The result thus follows from Lemma 8. $\square$

Nongeneric tangent lines

We count here the number of nongeneric tangent lines. Note that, as mentioned before, there are no such lines under some adequate general position assumption.

We first introduce some notation. A plane Π_t is called a *tangent plane* if and only if it is tangent to **P**, **Q**, or **R** and at least one point of tangency does not lie on l_e.

LEMMA 10. *A nongeneric tangent line lies in a tangent plane or properly intersects an edge of **P**, **Q**, or **R** lying in l_e.*

PROOF. Let ℓ be a nongeneric tangent line. By definition, ℓ properly intersects P_t, Q_t, or R_t in some plane Π_t. Suppose without loss of generality that ℓ properly intersects P_t. Since ℓ is tangent to **P**, P_t is a face or an edge of **P** which implies that Π_t is tangent to **P**. Thus Π_t is a tangent plane or P_t is an edge contained in l_e. In the first case, ℓ lies in the tangent plane Π_t. In the latter case, ℓ properly intersects an edge of **P** lying in l_e. $\square$

We count first the nongeneric tangent lines lying in tangent planes and then the ones that properly intersect an edge of **P**, **Q**, or **R** lying in l_e.

LEMMA 11. *The nongeneric tangent lines lying in all tangent planes form $O(p+q+r)$ connected components.*

PROOF. We first consider the nongeneric tangent lines lying in a given tangent plane Π_t. Such lines intersect, properly or not, e, P_t, Q_t, and R_t in the plane Π_t. Let ℓ denote such a line and $\mathcal{A}_p$ (resp. $\mathcal{A}_t$) the subset of $\{e, P_t, Q_t, R_t\}$ of polygons that properly intersect ℓ (resp. are tangent to ℓ). We can move ℓ in Π_t into a line tangent to at least two polygons of $\{e, P_t, Q_t, R_t\}$, while intersecting (properly ot not) the polygons of $\mathcal{A}_p$ and remaining tangent to the polygons of $\mathcal{A}_t$ during the motion. There are $O(p+q+r)$ such tangents, thus the nongeneric tangent lines lying in Π_t form $O(p+q+r)$ connected components.

The result follows since there are at most 2 tangent planes per polytope. $\square$

LEMMA 12. *The nongeneric tangent lines that properly intersect an edge of **P**, **Q**, or **R** lying in l_e form $O(p+q+r)$ connected components.*

PROOF. Let ℓ denote such a line. Let o_p be the union of edges of **P** that are contained in l_e; if no such edge exists, o_p is empty. We define o_q and o_r similarly. Line ℓ intersects at least one of o_p, o_q, and o_r. Let $\alpha = 1$, 2, or 3 be the number of these sets intersected by ℓ. We consider in turn the three cases depending on the value of α.

Case $\alpha = 3$. Sets o_p, o_q, and o_r are all nonempty since ℓ intersects each of them. If o_p, o_q, o_r and e do not all pairwise intersect, i.e., if $I = e \cap o_p \cap o_q \cap o_r$ is empty then there is exactly one line ℓ intersecting all four o_p, o_q, o_r and e, and that is line l_e. If on the contrary, I is nonempty, then the nongeneric tangent lines intersecting I are the line intersecting I and not intersecting the interior of **P**, **Q**, **R** and **S**. These lines thus form one connected component.

Case $\alpha = 2$. Suppose that ℓ intersects o_p and o_q; the other cases are similar. Then o_p and o_q are nonempty. Again if $I = e \cap o_p \cap o_q$ is empty, then there is at most one line $\ell = l_e$ intersecting e, o_p, and o_q and tangent to **R**. If on the contrary, I is nonempty then there are at most r edges of **R** that may induce some tangents with I. By Lemma 8, that implies that there are at most $O(r)$ connected components of lines intersecting e, o_p, and o_q, and tangent to **P**, **Q**, **R** and **S**.

Case $\alpha = 1$. Suppose that ℓ intersects o_p; the other cases are similar. Then o_p is nonempty. Again if $I = e \cap o_p$ is empty, then there is at most one line $\ell = l_e$ intersecting both o_p and e and tangent to polytopes **Q** and **R**. Consider now the case where the common intersection I is nonempty. Lemmas 9, 10 and 11, when applied to I and the three polytopes **Q**, **R**, and again **R** yields that there are at most $O(q+r)$ connected components of lines intersecting I and tangent to **P**, **Q**, **R** and **S**. $\square$

Lemmas 10, 11 and 12 yield the following result.

LEMMA 13. *There are at most $O(p+q+r)$ connected components of nongeneric tangent lines.*

This result together with Lemma 9 concludes the proof of the Main Lemma.

3. UPPER BOUND FOR THEOREM 2

We prove in this section the upper bound of Theorem 2 on the number of tangents to any 4 among k polytopes. The lower bound is proved in Section 4

Consider k polytopes $\mathbf{P}_1, \ldots, \mathbf{P}_k$ with $n_1, \ldots, n_k$ edges respectively. Let $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_l$, and $\mathbf{P}_m$ be four pairwise distinct polytopes and e an edge of $\mathbf{P}_i$. From the Main Lemma, we know that the number of connected components of lines intersecting e and tangent to $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_l$, and $\mathbf{P}_m$ is no more than $C(n_j + n_l + n_m)$, where C is some constant. There are n edges in the scene, so the number T of connected components of tangents to four polytopes satisfies

$$T \leqslant n \sum_{j<l<m} C(n_j + n_l + n_m).$$

Since each n_i, $1 \leqslant i \leqslant k$, appears $\binom{k-1}{2}$ times in the sum,

$$T \leqslant C n \sum_{1 \leqslant i \leqslant k} n_i \binom{k-1}{2} = C n^2 \binom{k-1}{2}$$

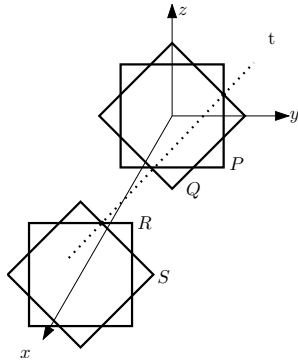so T is $O(n^2 k^2)$ as claimed.

Figure 7: Lower bound example for Lemma 14.

4. LOWER BOUNDS FOR THEOREMS 1 AND 2

We provide in this section the lower bounds needed for Theorems 1 and 2.

LEMMA 14. *There exist four disjoint polytopes of complexity $O(n)$ such that the number of common tangent lines is finite and $\Omega(n^2)$. There also exist two polytopes of complexity $O(n)$ and two polytopes of complexity $O(m)$ such that the number of common tangent lines is finite and $\Omega(mn)$.*

PROOF. We consider four planar regular polygons P, Q, R, and S, each with n vertices, embedded in $\mathbb{R}^3$. P is centered at the origin and parallel to the yz-plane, Q is obtained from P by a rotation of angle $\frac{\pi}{n}$ about the x-axis, and R and S are obtained from P and Q, respectively, by a translation of length 1 in the positive x-direction (see Figure 7). We transform the polygons P and Q into the polytopes **P** and **Q** by adding a vertex at coordinates $(\varepsilon, 0, 0)$. Similarly, we transform the polygons R and S into the polytopes **R** and **S** by adding a vertex at coordinates $(1 + \varepsilon, 0, 0)$.

For ε sufficiently small, the lines tangent to **P**, **Q**, **R** and **S** are the lines through a vertex of $P \cap Q$ and a vertex of $R \cap S$. Since $P \cap Q$ and $R \cap S$ have $2n$ vertices each, there are $4n^2$ tangent lines. Now, moving **P** and **S** by 2ε in the x direction ensures the disjointness of the polytopes while preserving the existence of the tangents if ε is small enough.

Replacing R and S in the above construction by regular polygons each with m vertices yields the $\Omega(mn)$ lower bound in the case of two polytopes of complexity $O(n)$ and two polytopes of complexity $O(m)$. □

We now provide the lower bound needed for Theorem 2.

LEMMA 15. *There exist k disjoint polytopes of total complexity $O(n)$ such that the number of non-occluded maximal line segments tangent to four of them is finite and $\Omega(k^2n^2)$.*

PROOF. The lower bound example is similar to the one with four polyhedra. For simplicity suppose that n and k are such that $\frac{n}{k}$ and $\frac{k}{4}$ are integers. We first take a $\frac{n}{k}$-regular polygon A_1 in the plane $x = 0$. Next we consider a copy, B_0, of A_1 scaled by a factor of $(1 + \varepsilon)$, and on each edge of B_0 we place $\frac{k}{4}$ points. Polygon B_i, $1 \leqslant i \leqslant \frac{k}{4}$, is constructed by taking the ith point on each edge of B_0. If ε is small enough, the intersection points of A_1 and B_i are outside the other polygons B_j for $1 \leqslant j \leqslant \frac{k}{4}$ and $i \neq j$. Now the A_i, for $2 \leqslant i \leqslant \frac{k}{4}$, are constructed as copies of A_1 scaled by a factor
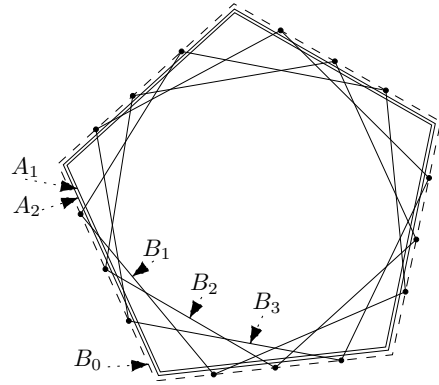

Figure 8: The polygons A_i and B_i.

$1 + \frac{i}{k}\varepsilon$ (see Figure 8). For the moment, all polygons lie in plane $x = 0$. We now construct 4 families of $\frac{k}{4}$ polygons each:
- P_i is a copy of A_i translated by $i\epsilon$ in the negative x direction
- Q_i is a copy of B_i translated by $i\epsilon$ in the positive x direction
- R_i is a copy of B_i translated by $1 - i\epsilon$ in the positive x direction
- S_i is a copy of A_i translated by $1 + i\epsilon$ in the positive x direction

Any choice of four polygons, one in each family P_i, Q_j, R_l and S_m, reproduces the quadratic example of Lemma 14 with polygons of size $\frac{n}{k}$ and thus with total number of tangents larger than $\left(\frac{k}{4}\right)^4 4 \left(\frac{n}{k}\right)^2 = \frac{n^2 k^2}{4}$. Furthermore the lines tangent to P_i, Q_j, R_l and S_m are only occluded by $P_{i'}$ and $S_{m'}$ for $i' > i$ and $m' > m$, that is, beyond the portion of the tangents containing the contact points. The k polygons can be transformed into k convex polyhedra as in Lemma 14. □

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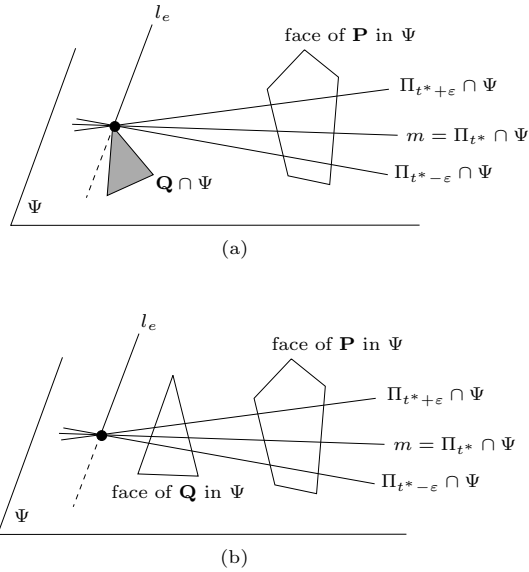
Figure 9: m **is tangent to P along a face in** Ψ **and (a) to** $\mathbf{Q} \cap \Psi$ **only on** l_e **or (b) to Q along a face in** Ψ.

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APPENDIX

Proof of Claim E

We give here a complete proof of Claim E of Lemma 5 when the polytopes are in arbitrary position. We first recall the claim.

CLAIM E. *Line m is tangent to* $\mathbf{P} \cap \Psi$ *or to* $\mathbf{Q} \cap \Psi$, *at some point not on* l_e.

In the following, we keep all the notation and assumptions made in the proof of Lemma 5 and all the claims that are referred to are those of Lemma 5.

We assume for a contradiction that line m does not satisfy Claim E, i.e., m is not tangent to $\mathbf{P} \cap \Psi$ or to $\mathbf{Q} \cap \Psi$ at any point other than on l_e. We prove that the set of supports of m is u and is a maximal (P_t, Q_t)-tuple for all t in any sufficiently small neighborhood of t^*, contradicting the hypotheses of Lemma 5.

Since m is tangent to $\mathbf{Q}$ (by Claim C), m is tangent to $\mathbf{Q} \cap \Psi$ only on l_e (see Figure 9(a)), or m properly intersects $\mathbf{Q} \cap \Psi$ which is then a face or an edge of $\mathbf{Q}$ (see Figure 9(b)). Similarly m is tangent to $\mathbf{P} \cap \Psi$ only on l_e, or m properly intersects it; however $\mathbf{P} \cap \Psi$ is necessarily a face of $\mathbf{P}$ by Claim D.

Let m_t be the line $\Psi \cap \Pi_t$ for all t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*; line m_t is well defined since $\Psi \cap \Pi_{t^*}$ is line m by Claims C and D. Recall that we only consider here supports in $\mathbf{P}$ and $\mathbf{Q}$.

LEMMA 16. *The set of supports of m_t is u for some t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*.*

PROOF. We first prove that the supports in u are supports of m_t for all $t \in \mathcal{N}$. A support vertex in u lies on l_e by Claim A and thus lies in Π_t for all t. A support vertex in u also lies on m by Claim C and thus lies in plane Ψ by Claim D. Hence, for all $t \in \mathcal{N}$, the support vertices in u lie on m_t, and thus are supports of m_t.

In order to prove that the support edges in u are supports of m_t, it is sufficient (by Claim C) to prove that the support edges of m are supports of m_t. The support edges of m in $\mathbf{P}$ lie in plane Ψ (see Figure 9(b)) because Ψ contains m and a face of $\mathbf{P}$ (indeed if m intersects an edge of $\mathbf{P}$ not in Ψ then m contains one of its endpoints, and thus the edge is not a support). Thus all the support edges of m lie in Ψ and m contains none of their endpoints (by definition). Since m_t lies in Ψ for all t and $m_{t^*} = m$, line m_t intersects all the support edges of m and contains none of their endpoints for all t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*. Hence the support edges of m in $\mathbf{P}$ are supports of m_t for all $t \in \mathcal{N}$.

Consider the case where $\mathbf{Q} \cap \Psi$ is a face or an edge of $\mathbf{Q}$. Similarly as for $\mathbf{P}$, the support edges of m in $\mathbf{Q}$ lie in plane Ψ, and thus are supports of m_t for all $t \in \mathcal{N}$.

Consider now the case where m is tangent to $\mathbf{Q} \cap \Psi$ only on l_e at, say, point v (see Figure 9(a)). Then v lies in Ψ (since $m \subset \Psi$ by Claim D) and also lies in Π_t for all t (since $l_e \subset \Pi_t$ for all t). Hence m_t contains v for all $t \in \mathcal{N}$. Moreover, m_t is tangent to $\mathbf{Q} \cap \Psi$ only at v for all t in any sufficiently small open neighborhood $\mathcal{N}$ of t^*. Hence the set of supports of m_t in $\mathbf{Q}$ is invariant for all $t \in \mathcal{N}$.

We have so far proved that the set of supports of m_t contains u for all $t \in \mathcal{N}$.

We now prove that the set of supports of m_t is u for some $t \in \mathcal{N}$. Consider first the case where u is a maximal (P_{t^*}, Q_{t^*})-tuple. Then, by Claim A, the union of the supports in u intersects Π_{t^*} in at least two distinct points, thus $m_{t^*} = m$ is the only line in Π_{t^*} whose set of supports contains u. Moreover, since u is a (P_{t^*}, Q_{t^*})-tuple, there exists a line in Π_{t^*} whose set of supports is u. Hence the set of supports of m_{t^*} is u.

Consider now the case where u is a maximal (P_t, Q_t)-tuple for all $t \in (t^*, t_1)$. By Claim A, for all $t \in (t^*, t_1)$, the union of the supports in u intersects Π_t in at least two distinct points, thus m_t is the only line in Π_t whose set of supports contains u. For all $t \in (t^*, t_1)$, since u is a (P_t, Q_t)-tuple there exists a line in Π_t whose set of supports is u. Hence the set of supports of m_t is u for all $t \in (t^*, t_1)$. $\square$

LEMMA 17. *The set of supports of m_t is invariant for t ranging in any sufficiently small open neighborhood $\mathcal{N}$ of t^*.*

PROOF. First if $m = l_e$ then $m_t = l_e$ for all $t \in \mathcal{N}$ because Ψ contains $m = l_e$ (by Claim D) and Π_t contains l_e for all t (by definition). Thus the set of supports of m_t is invariant for all $t \in \mathcal{N}$. We now assume that $m \neq l_e$.

Line m is tangent to polygon P_{t^*} along an edge by Claim C. Thus m is tangent to $\mathbf{P}$ in at least two points. Hence, since $\mathbf{P} \cap \Psi$ is a face of $\mathbf{P}$ and m lies in Ψ, either m properly intersects $\mathbf{P} \cap \Psi$ or m is tangent to $\mathbf{P} \cap \Psi$ along one of its edges. In the later case, the edge does not lie in l_e since $m \neq l_e$, thus m is tangent to $\mathbf{P} \cap \Psi$ at some point not on l_e, contradicting our assumptions. Hence m properly intersects the face of $\mathbf{P}$ in Ψ.

It follows that, if m contains a vertex of $\mathbf{P}$, then this vertex is an endpoint of a support edge of m_t for all t in any sufficiently small open neighborhood of t^* (indeed m_t lies in Ψ and tends to m when t tends to t^*). By Lemma 16, the set of supports of m_t is u for some t in any sufficiently small open neighborhood of t^*. Hence, if m contains a vertex of $\mathbf{P}$, this vertex is an endpoint of a support edge in u. By assumption u contains no edge with endpoint on $\Pi_{t^*} \setminus l_e$, thus m contains no vertex of $\mathbf{P}$ except possibly on l_e (since m lies in Π_{t^*}). It thus follows that the set of supports of m_t in $\mathbf{P}$ is invariant for t ranging in any sufficiently small open neighborhood of t^* (since $m_t \subset \Psi$ tends to m when t tends to t^* and all supports of m lie in Ψ).

Now consider the case where m properly intersects $\mathbf{Q} \cap \Psi$ which is a face or an edge of $\mathbf{Q}$. Similarly as for $\mathbf{P}$, m contains no vertex of $\mathbf{Q}$ except possibly on l_e and thus the set of supports of m_t in $\mathbf{Q}$ is invariant for t ranging in any sufficiently small open neighborhood of t^*.

Finally, consider the case where m is tangent to $\mathbf{Q} \cap \Psi$ only on l_e. Then, as in the proof of Lemma 16, the set of supports of m_t in $\mathbf{Q}$ is invariant for all t ranging in any sufficiently small open neighborhood of t^*, which concludes the proof. □

PROOF OF CLAIM E. It follows from Lemmas 16 and 17 that the set of supports of m_t is invariant and equal to u for all $t \in \mathcal{N}$. Moreover, since m_t varies continuously with t and $m = m_{t^*}$ is tangent to P_{t^*} and Q_{t^*} (by Claim C), line m_t is tangent to P_t and Q_t for all $t \in \mathcal{N}$. Hence u is a (P_t, Q_t)-tuple for all $t \in \mathcal{N}$. We now prove that u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}$.

As we have seen before, $m = m_{t^*}$ is tangent to $\mathbf{P}$ in at least two points (by Claim C), thus m_{t^*} intersects its supports in at least two distinct points. Moreover the set of supports of m_{t^*} is u. Thus there is a unique line in Π_{t^*} whose set of supports contains u. Hence u is a maximal (P_{t^*}, Q_{t^*})-tuple.

By Claim A, m_t is the unique line in Π_t whose set of supports contains u for all $t \neq t^*$ in $\mathcal{N}$. Thus u is a maximal (P_t, Q_t)-tuple for all $t \neq t^*$ in $\mathcal{N}$.

Hence u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}$, contradicting the hypotheses of Lemma 5 and thus concluding the proof of Claim E. □
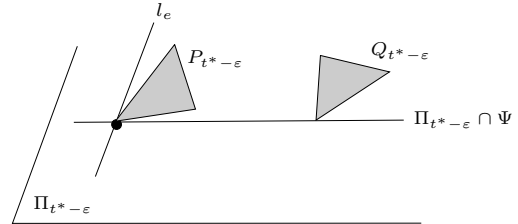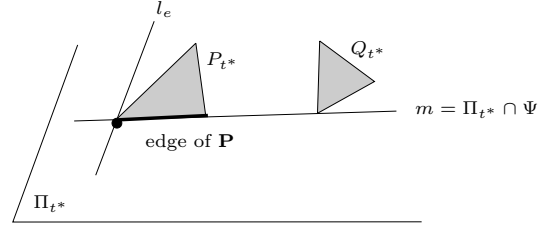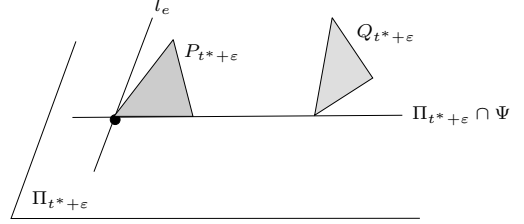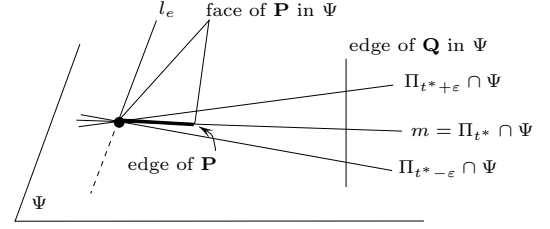

Figure 10: The set u of supports of line $\Pi_{t^*-\varepsilon} \cap \Psi$ is a maximal (P_t, Q_t)-tuple for some but not all t in any open neighborhood of t^*. There is a F-critical event (t^*, m) (where $m = \Pi_{t^*} \cap \Psi$) such that u is contained in the set of supports of m because (i) m lies in plane $\Psi \neq \Pi_{t^*}$ containing a face of $\mathbf{P}$ and (ii) m is tangent to $\mathbf{P} \cap \Psi$ at some point not on l_e. Note that there is no V-critical event (t^*, v) such that u contains v or an edge with endpoint v, and there is no line $\ell \in \Pi_{t^*}$ such that (i) ℓ lies in a plane $\Upsilon \neq \Pi_{t^*}$ containing a face of P and (ii) ℓ is tangent to $\mathbf{Q} \cap \Upsilon$ at some point not on l_e; indeed ℓ is necessarily equal to m, if $\Upsilon = \Psi$ then ℓ is not tangent to $\mathbf{Q} \cap \Upsilon$ and we can consider that the plane, distinct from Ψ, that contains ℓ and a face of $\mathbf{P}$ is equal to Π_{t^*}.